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LETTER FROM THE EDITOR-IN-CHIEF

Farid AitSahlia
Warrington College of Business,
University of Florida

Generalizations of the celebrated Black–Scholes formula have been pursued in a variety of directions. Variations that account for stylized facts such as stochastic volatility and jumps have resulted in statistical estimation challenges. This issue of The Journal of Risk evaluates option pricing alternatives, particularly nonparametric models and those developed ad hoc by practitioners. Other topics of practical importance, namely the impact of high-frequency trading on parameter estimation and the optimal execution of share repurchases by firms, are also considered in this issue.

In the first paper, “Pricing and hedging options with rollover parameters”, Sol Kim contrasts rollover strategies to update parameter estimates and their impact on pricing and hedging models. Using Standard & Poor’s 500 data, Kim shows that the so-called absolute smile ad hoc modification of the Black–Scholes model is simple to implement, and tends to provide pricing and hedging results that are more accurate than extension models accounting for stochastic volatility and jumps.

These extension models are also assessed by Xiaolong Zhong, Jie Cao, Yong Jin and Wei Zheng in “On empirical likelihood option pricing”, the second paper in this issue. Here, through Standard & Poor’s data and simulation, the authors implement an empirical likelihood model, which they show also generates more accurate prices than classic nonparametric and stochastic volatility with jumps models.

Ben Charoenwong and Guanhao Feng contrast the long-horizon volatility forecasting accuracy of high- and low-frequency data in “Does higher-frequency data always help to predict longer-horizon volatility?”. Their paper, the third in this issue, shows that mean model misspecification leads to a deterioration in the quality of volatility estimates as data frequency increases, especially in the presence of high conditional autocorrelation.

In our fourth and final paper, “Optimal execution of accelerated share repurchase contracts with fixed notional”, Olivier Guéant considers the problem of a firm that wants to repurchase a subset of its own shares while only making a minimal impact on their price. Guéant proposes a discrete dynamic programming model to determine the best price for the accelerated share repurchase (ASR) contract that the firm arranges with its investment bankers, as well as its optimal execution. This approach highlights certain features of ASR contracts: namely, that they involve issues related to optimal trade execution as well as the pricing of Asian and Bermudan options.
Research Paper

Pricing and hedging options with rollover parameters

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ABSTRACT

We implement a “horse race” competition between several option-pricing models for Standard & Poor’s 500 options. We consider trader rules (the so-called ad hoc Black–Scholes model) to predict future implied volatilities by applying simple ad hoc rules, as well as mathematically complicated option-pricing models, to the observed current implied volatility patterns. The traditional rollover strategy, ie, the nearest-to-next approach, and a new rollover strategy, the next-to-next approach, are also compared for the parameters of each option-pricing model. We find that simple trader rules dominate mathematically more sophisticated models, and that the next-to-next strategy can decrease the pricing and hedging errors of all option-pricing models, unlike the nearest-to-next approach. The “absolute smile” trader rule, which assumes that the implied volatility follows a fixed function of the strike price, has the advantage of simplicity and is the best model for pricing and hedging options.

Keywords: options pricing; volatility smile; trader rules; stochastic volatility; jumps.

1 INTRODUCTION

Ever since Black and Scholes published their seminal article on options pricing (Black and Scholes 1973), various theoretical and empirical studies on options pricing have

Bakshi et al (1997, 2000) and Kim and Kim (2005) conduct a comprehensive empirical study on the relative merits of competing option-pricing models. They discover that taking SV into account is of first-order importance in improving the BS model. However, Dumas et al (1998), Jackwerth and Rubinstein (2012), Li and Pearson (2007) and Kim (2009) examine the performance of a number of these mathematically sophisticated models and, among striking empirical findings, show that they predict option prices less well than a pair of ad hoc approaches sometimes used by option traders. These ad hoc approaches can be an alternative to complicated models for pricing options and are known as ad hoc BS (AHBS) models.

There are two versions of the AHBS model. In the “relative smile” approach, the implied volatility skew is treated as a fixed function of moneyness, \( S/K \), where the implied volatility for a fixed strike price, \( K \), varies with the spot price, \( S \). In the “absolute smile” approach, the implied volatility is treated as a fixed function of the strike price, and for a fixed strike price it does not vary with the spot price. Jackwerth and Rubinstein (2012), Li and Pearson (2007), Kim (2009), Choi et al (2012) and Choi and Ok (2012) find that the “absolute smile” approach performs better than the “relative smile” approach for pricing options. Furthermore, the simpler version of the AHBS model performs better than the other model. That is, the presence of more parameters in AHBS models actually causes overfitting.

When the options are priced and hedged, we need to estimate the parameters that we need to plug into each model. For one-day-ahead pricing and hedging, the parameters are estimated using the previous day’s option data. For one-week-ahead pricing and hedging, the option data from the preceding seven days is used. There are no complications with general trading dates. However, it is standard to eliminate the nearest option contracts that expire in under seven days because of the liquidity issue. When six days remain to maturity, the nearest option contracts are removed and the next-to-nearest option contracts are added to the sample. When forecasting the parameters for the next-to-nearest option contracts on that day, the problem of finding suitable rollover strategies for the parameters arises: to estimate the parameters for the next-to-nearest contract on day \( t \), we can use the option data for the nearest term contract.
on day $t - 1$ (nearest-to-next rollover strategy), or we can use the next-to-nearest contract on day $t - 1$ (next-to-next rollover strategy).\footnote{This explains the one-day-ahead out-of-sample pricing and hedging cases. The same methodology can be applied to one-week-ahead cases.} Choi and Ok (2012) showed that the next-to-next rollover strategy can mitigate the overfitting problems of AHBS models and allows an AHBS model with more parameters to perform better than an AHBS model with fewer parameters. Therefore, the next-to-next rollover strategy can be useful for AHBS-type models.

Is the next-to-next strategy effective only for AHBS models, however, or can it also be applied to mathematically complicated models, such as SV and SV with jump (SVJ) models? In this paper, we examine the empirical performance of several option-pricing models with respect to the parameter strategies. Both trader rules, ie, both AHBS-type models and SV and SVJ models, are considered for a “horse race” competition. We compare the pricing and hedging performance of several option-pricing models using the traditional parameter rollover strategy, ie, the nearest-to-next approach, with models using the new strategy, ie, the next-to-next approach. We examine whether the new rollover strategy can be effective not only for AHBS-type models but also for mathematically complicated models, such as the SV and SVJ models. After considering the new rollover strategy, we aim to determine the best option-pricing model.

To the best of our knowledge, this is the first study to exhaustively examine the performance of the rollover strategies for various option-pricing models. We shall fill in the following gaps that have been left open in previous research.

First, previous studies examining rollover strategies do not consider mathematically complicated models, which have been shown to be competitive option-pricing models. In this paper, we examine whether the new rollover strategy is effective for the SV and SVJ models.

Second, in previous research the next-to-next strategy is not considered in hedging performance. When we try to determine the best option-pricing model, both pricing and hedging performance must be considered. The pricing performance of the option-pricing models measures the ability to forecast the option price level; however, hedging performance measures the ability to forecast the variability of options prices. If a specific model performs better than the other models for both performance measures, then it is truly the best option-pricing model.

Third, previous studies consider a sample period of just two or three years. For example, for a two-year sample, there are twenty-four days that require a rollover of the parameters when we examine the one-day-ahead out-of-sample pricing and hedging performance. The effect of the rollover strategy can be exaggerated by a small sample. In this paper, we examine the rollover strategies using sample dates
spanning thirteen years. If the new rollover strategy works well, even for a long sample period, we can conjecture a structural change in the parameters when the maturity of options rolls over.

Fourth, recent research that examines the performance of AHBS-type models has considered Korea Composite Stock Price Index (KOSPI) 200 options, which are one of the emerging markets. We use the Standard & Poor’s 500 (SPX) option prices for our empirical work. S&P 500 options have been the focus of many investigations, including those of Dumas et al (1998), Bakshi et al (1997), Jackwerth and Rubinstein (2012) and Li and Pearson (2007). In addition, the rollover strategies for the parameters are not compared in the SPX options market. If the new rollover strategy is well implemented for SPX options, we can conjecture that it not only is fit for emerging markets, but also can be generally applied to advanced options markets.

We find that simple trader rules dominate more mathematically sophisticated models, and the new rollover strategy (the next-to-next strategy) can decrease the pricing and hedging errors of all option-pricing models, unlike the nearest-to-next approach. When the next-to-next strategy is considered, AHBS-type models with more parameters perform better than both the ad hoc approaches, which have fewer parameters, and the mathematically complicated models for both pricing and hedging options. That is, the next-to-next strategy can mitigate the overfitting problem of AHBS-type models. The “absolute smile” trader rule, which assumes that the implied volatility follows a fixed function of the strike price, has the advantage of simplicity and is the best model for pricing and hedging options.

This paper is organized as follows. Section 2 reviews the AHBS-type models, the SVJ model and the rollover strategies for the parameters. Section 3 describes the data. Section 4 describes the parameter estimates of each model and evaluates the pricing and hedging performance of alternative option-pricing models. Section 5 concludes our study and summarizes the results.

2 OPTIONS PRICING MODELS

2.1 AHBS model

Despite its significant pricing and hedging biases, the BS model continues to be widely used by market practitioners. However, when practitioners apply the BS model, they commonly allow the volatility parameter to vary across option strike prices and fit the volatility to the observed smile pattern. As Dumas et al (1998) show, this procedure can avoid some of the biases associated with the BS model’s constant volatility assumption.

We need to construct an AHBS model in which each option has its own implied volatility depending on the strike price (or moneyness) and time to maturity, or a
combination of both. However, we only consider the function of the strike price (or moneyness), because the liquidity of the index options market is concentrated in the nearest expiration contract. Dumas et al (1998) show that the specification including a time parameter performs worst of all, indicating that the time variable is an important cause of the overfitting problem at the estimation stage.

There are two versions of the ad hoc approach. In the “relative smile” approach, the implied volatility skew is treated as a fixed function of moneyness, $S/K_i$, and the implied volatility for a fixed strike price, $K_i$, varies as the spot price, $S$, varies. This is also known as the sticky volatility method. In the “absolute smile” approach, the implied volatility is treated as a fixed function of the strike price, $K_i$, and the implied volatility for a fixed strike price does not vary with the spot price, $S$. This is also called the sticky delta method. These models are the so-called AHBS models. Dumas et al (1998), Jackwerth and Rubinstein (2012), Li and Pearson (2007), Kim (2009), Choi et al (2012) and Choi and Ok (2012), who report that the AHBS model outperforms other option-pricing models, adopt the “absolute smile” approach. On the other hand, Kirgiz (2001) and Kim and Kim (2004), who report that the AHBS model does not outperform others, adopt the “relative smile” approach. That is, the specific type of AHBS model seems to be important for pricing and hedging performance.

Specifically, we adopt the following six specifications for the BS implied volatilities:

\[
\sigma_i = \beta_1 + \beta_2 (S/K_i), \quad (R1)
\]
\[
\sigma_i = \beta_1 + \beta_2 (S/K_i) + \beta_3 (S/K_i)^2, \quad (R2)
\]
\[
\sigma_i = \beta_1 + \beta_2 (S/K_i) + \beta_3 (S/K_i)^2 + \beta_4 (S/K_i)^3, \quad (R3)
\]
\[
\sigma_i = \beta_1 + \beta_2 K_i, \quad (A1)
\]
\[
\sigma_i = \beta_1 + \beta_2 K_i + \beta_3 K_i^2, \quad (A2)
\]
\[
\sigma_i = \beta_1 + \beta_2 K_i + \beta_3 K_i^2 + \beta_4 K_i^3, \quad (A3)
\]

where $\sigma_i$ is the implied volatility for the $i$th option of strike $K_i$ and spot price $S$.

The R1, R2 and R3 models are “relative smile” approaches using relative moneyness as the independent variable. The A1, A2 and A3 models are “absolute smile” approaches using the absolute strike prices as the independent variables. R1 is an AHBS model that considers the intercept and moneyness as the independent variables; R2 is an AHBS model that considers the intercept, moneyness and moneyness squared as the independent variables; and R3 is an AHBS model that considers the intercept, moneyness, moneyness squared and the third power of moneyness as the independent variables. A1 is an AHBS model that considers the intercept and the strike price as the independent variables; A2 is the AHBS model that considers the intercept, the strike price and the strike price squared as the independent variables;
and A3 is the AHBS model that considers the intercept, the strike price, the strike price squared and the third power of the strike price as the independent variables.

For the implementation, we follow a four-step procedure. First, we abstract the BS implied volatilities from each option. Second, we set up the implied volatilities as the dependent variable and moneyness or the strike price as the independent variables. We also estimate $\hat{\beta}_i$ ($i = 1, 2, 3, 4$) by ordinary least squares (OLS). Third, using the parameters estimated in the second step, we plug each option’s moneyness or strike price into the equation and obtain its model-implied volatility. Finally, we use the volatility estimates computed in the third step to price options with the following BS formulas:

$$C_{t,\tau} = S_t N(d_1) - Ke^{-r\tau} N(d_2),$$

$$P_{t,\tau} = Ke^{-r\tau} N(-d_2) - S_t N(-d_1),$$

$$d_1 = \frac{\ln[S_t/K] + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau},$$

where $N(\cdot)$ is the cumulative standard normal density. The AHBS model, although theoretically inconsistent, can be a more challenging benchmark than the simple BS model for any competing option valuation model.

## 2.2 SVJ model

Bakshi et al (1997, 2000) and Kim and Kim (2005) conducted a comprehensive empirical study on the relative merits of competing option-pricing models. They found that taking SV into account is of first-order importance in improving the BS model, and jumps can be useful for pricing short-term options. Bakshi et al (1997) derived a closed-form option-pricing model that incorporates both SV and jumps. Under the risk-neutral measure, the underlying non-dividend-paying stock price $S_t$ and its components for any time $t$ are given by

$$\frac{dS_t}{S_t} = [r - \lambda J_t] dt + \sqrt{v_t} dz_{S,t} + J_t dq_t,$$

$$dv_t = [\theta_v - \kappa v_t] dt + \sigma_v \sqrt{v_t} dz_{v,t},$$

$$\ln[1 + J_t] \sim \mathcal{N}\left(\ln[1 + \mu_J] - \frac{1}{2\sigma^2_J}, \sigma^2_J\right),$$

where $r$ is the instantaneous spot interest rate; $\lambda$ is the frequency of jumps per year; $v_t$ is the diffusion component of the return variance (conditional on no jumps); $z_{S,t}$ and $z_{v,t}$ are standard Brownian motions, with $\text{Cov}_t[dz_{S,t}, dz_{v,t}] = \rho dt$; and $J_t$ is the percentage jump size (conditional on a jump) that is lognormally, identically and independently distributed over time, with unconditional mean $\mu_J$. The standard
deviation of $\ln(1 + J_t)$ is $\sigma_J$; $q_t$ is a Poisson jump counter with intensity $\lambda$, ie, $\Pr[ dq_t = 1 ] = \lambda \, dt$; and $\Pr[ dq(t) = 0 ] = 1 - \lambda \, dt$. The terms $\kappa_v$, $\theta_v/\kappa_v$ and $\sigma_v$ are the speed of adjustment, the long-run mean and the variation coefficient of the diffusion volatility, $v_t$, respectively. The variables $q_t$ and $J_t$ are uncorrelated with each other or with $z_{S,t}$ and $z_{v,t}$.

For a European call option written on a stock with strike price $K$ and time to maturity $\tau$, the closed-form formula for price $C_{t,\tau}$ at time $t$ is

$$
C_{t,\tau} = S_t \, P_1(t, \tau; S_t, r, v_t) - K \, e^{-r_t \tau} \, P_2(t, \tau; S_t, r, v_t),
$$

(2.7)

where the risk-neutral probabilities $P_1$ and $P_2$ are computed by inverting the respective characteristic functions of the following equation:

$$
P_j(t, \tau; S_t, r, v_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{\exp(-i\varphi \ln K) \, f_j(t, \tau; S_t, r, v_t; \varphi)}{i\varphi} \right],
$$

(2.8)

The characteristic functions, $f_j$, are given in (A-1) and (A-2) of the online appendix. The price of a European put on the same stock can be determined from put–call parity. The option valuation model in (2.7) and (2.8) contains the most models as special cases. For example, we obtain

(i) the BS model by setting $\lambda = 0$, $\kappa_v = \theta_v/\kappa_v = 0$, and

(ii) the SV model by setting $\lambda = 0$, where L’Hôpital’s rule may be needed to derive each special case from (2.8).

In applying the option-pricing models, we always encounter the difficulty of unobservable spot volatilities and structural parameters. As in standard practice, we estimate the parameters of each model for every sample day. Since closed-form solutions are available for an option price, a natural candidate for the estimation of the parameters in the formula is a nonlinear least squares procedure involving minimization of the sum of squared errors between the model and the market prices:

$$
\min_{\varphi} \sum_{i=1}^N (O_{i,t} - O_{i,t}^*)^2,
$$

(2.9)

2 The objective function to minimize the sum of percentage squared errors can be used. However, in this case, the AHBS models are calibrated by OLS on implied volatilities, whereas the SV and SVJ models are calibrated on relative pricing errors. Christoffersen and Jacobs (2004) address such inconsistencies in the choice of objective function and argue that the results from the mixture of the two different objectives have little explanatory power. In particular, they argue that the in-sample objective function must be the same as the out-of-sample objective function. In addition, in our sample there is no great difference between the results using the sum of the squared errors and those using the sum of the percentage squared errors.
where, for option $i$ on day $t$, $O_{i,t}^*$ denotes the model price, $O_{i,t}$ denotes the market price, $N$ denotes the number of options and $\varphi_t$ denotes the parameters for each model. For the BS model, we estimate the volatility parameter $\sigma_t$. For the SV model, we estimate the variance parameter $v_t$ with structural parameters $\{\theta_v, \kappa_v, \rho, \sigma_v\}$. For the SVJ model, we estimate the variance parameter $v_t$ with structural parameters $\{\lambda, \mu_J, \sigma_J, \theta_v, \kappa_v, \rho, \sigma_v\}$. As mentioned above, the coefficients for the AHBS model are estimated via OLS.

An alternative method is to estimate the parameters from the asset returns. However, historical data reflects only what happened in the past. Furthermore, the procedure using historical data is not capable of identifying risk premiums, which must be estimated from the option data conditional on the estimates of other parameters. The important advantage of using option prices to estimate parameters is that it allows us to use the forward-looking information contained in the option prices.

### 2.3 Rollover strategies

When we price and hedge options, the parameters that are input into each model need to be estimated. As in standard practice, we estimate the parameters of each model using the option data for every sample day. For one-day-ahead pricing and hedging performance, the parameters are estimated using the previous day’s option data. For one-week-ahead pricing and hedging performance, option data from seven days ago is used. It is standard to eliminate the nearest option contracts that expire in under seven days because of the liquidity issue. When six days remain to maturity, the nearest option contracts are removed and the next-to-nearest option contracts are added to the sample. To estimate the parameters for the next-to-nearest contract on day $t$, we can use the option data for the nearest term contract on day $t - 1$ (nearest-to-next rollover strategy), whereas on day $t - 1$ we can use the next-to-nearest contract (next-to-next rollover strategy). The information content of these two contracts may differ, and therefore the rollover procedure may be important for the accuracy of parameter forecasting.

Figure 1 shows the difference between the nearest-to-next and the next-to-next strategies. It represents our example for one-day-ahead pricing and hedging performance. The circle represents the option data from the nearest option contract. The diamond represents the option data from the next-to-nearest option contract. To price and hedge the new next-to-nearest option contract, we use the previous day’s nearest term contract data for the nearest-to-next rollover strategy. When we consider the next-to-next rollover strategy, the parameters are estimated from the previous day’s next-to-nearest term contract data.

This paper compares the performances of the nearest-to-next and next-to-next strategies. If the next-to-next rollover strategy performs better than the nearest-to-next
strategy, we can conjecture structural change when the nearest-to-next contracts are changed to the nearest contracts.

3 DATA

Our S&P 500 index option data is from Option Metrics LLC. The data includes end-of-day bid and ask quotes, implied volatilities, open interest and daily trading volumes for the SPX options traded on the Chicago Board Options Exchange from January 4, 1996, through December 31, 2008. The data also includes daily index values and estimates of dividend yields, as well as the term structures of zero-coupon interest rates constructed from London Interbank Offered Rate quotes and eurodollar futures prices. We use the bid–ask average as our option price measure.

The following rules are applied to filter the data needed for the empirical test. We use out-of-the-money (OTM) options for calls and puts. First, since there is only a very thin trading volume for in-the-money (ITM) options, the credibility of the price information is not entirely satisfactory. Therefore, we use the price data regarding both put and call options that are near-the-money and OTM. Second, if we use both call and put option prices, to estimate the parameters we use ITM calls and OTM puts, which are equivalent according to put–call parity. Third, as Huang and Wu (2004, p. 1407) mention, “the Black–Scholes model has been known to systematically misprice equity index options, especially those that are out-of-the-money”. We recognize the need for an alternative option-pricing model to mitigate this effect.
This table reports the average option price and the number of options for each moneyness and options type (call or put) category. The sample period is from January 4, 1996 to December 31, 2008. The last bid–ask average of each option contract is used to obtain the summary statistics. Moneyness of an option is defined as $S/K$, where $S$ denotes the spot price and $K$ denotes the strike price.

Since options with less than seven days to expiration can induce biases due to low prices and bid–ask spreads, they are excluded from the sample. As mentioned before, this is why rollover strategies can arise. To mitigate the impact of price discreteness on option valuation, prices lower than 0.4 are omitted. Prices that do not satisfy the arbitrage restriction in the following equations are also excluded:

\[ S_t \geq C_{t,\tau} \geq S_t - Ke^{-r\tau}, \]  
\[ Ke^{-r\tau} \geq P_{t,\tau} \geq Ke^{-r\tau} - S_t, \]

where $C_{t,\tau}$ and $P_{t,\tau}$ are the call and put option prices, respectively, with strike price $K$ and time to maturity $\tau$ on day $t$, and $S_t$ and $r$ are the stock index and risk-free interest rate on day $t$, respectively.

We divide the option data into several categories, according to moneyness, $S/K$. Table 1 describes certain sample properties of the SPX option prices used in this study. Summary statistics are reported for the option prices, as well as for the total number of observations, according to each moneyness and option-type category. Note that there are 42,396 call and 64,316 put option observations, with deep OTM options comprising 24% for calls and 49% for puts.\(^3\) Table 2 presents the volatility smile (or “sneer”) effects for twenty-six consecutive six-month subperiods. We employ six fixed degree-of-moneyness intervals and compute the mean over alternative subperiods of the implied volatility. The SPX options market seems to be sneer independent in the subperiods employed in the estimation. As the $S/K$ ratio increases, the implied volatilities decrease to near-the-money; however, after the decrease, they increase to OTM put options. The implied volatility of deep OTM puts is greater than that of

\(^3\) For the call option, deep OTM options are options where $S/K < 0.94$. For the put option, deep OTM options are options where $S/K > 1.06$. 

\[ \text{Total} \]

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Call options</th>
<th></th>
<th>Put options</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>Number</td>
<td>Price</td>
<td>Number</td>
</tr>
<tr>
<td>$S/K &lt; 0.94$</td>
<td>2.7833</td>
<td>10,033</td>
<td>1.00 $&lt; S/K &lt; 1.03$</td>
</tr>
<tr>
<td>$0.94 &lt; S/K &lt; 0.96$</td>
<td>4.8054</td>
<td>13,580</td>
<td>$1.03 &lt; S/K &lt; 1.06$</td>
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<tr>
<td>$0.96 &lt; S/K &lt; 1.00$</td>
<td>12.2916</td>
<td>18,783</td>
<td>$S/K &gt; 1.06$</td>
</tr>
<tr>
<td>Total</td>
<td>7.6435</td>
<td>42,396</td>
<td>Total</td>
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</table>

Table 1: S&P 500 options data.
### TABLE 2  Implied volatility sneers.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
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<tbody>
<tr>
<td>1996 01-06</td>
<td>0.1308</td>
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<td>0.1207</td>
<td>0.1553</td>
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<td>0.2249</td>
</tr>
<tr>
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<td>0.1243</td>
<td>0.1289</td>
<td>0.1606</td>
<td>0.1884</td>
<td>0.2332</td>
</tr>
<tr>
<td>1997 01-06</td>
<td>0.1626</td>
<td>0.1579</td>
<td>0.1636</td>
<td>0.1895</td>
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<td>0.2516</td>
</tr>
<tr>
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<td>0.1957</td>
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<td>0.3092</td>
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<tr>
<td>1998 01-06</td>
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<td>0.1491</td>
<td>0.1588</td>
<td>0.1917</td>
<td>0.2226</td>
<td>0.2872</td>
</tr>
<tr>
<td>1998 07-12</td>
<td>0.2364</td>
<td>0.2037</td>
<td>0.2139</td>
<td>0.2406</td>
<td>0.2709</td>
<td>0.3492</td>
</tr>
<tr>
<td>1999 01-06</td>
<td>0.1831</td>
<td>0.1851</td>
<td>0.2011</td>
<td>0.2267</td>
<td>0.2511</td>
<td>0.3132</td>
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<tr>
<td>1999 07-12</td>
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<td>0.2417</td>
<td>0.3025</td>
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<tr>
<td>2000 07-12</td>
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<td>0.1891</td>
<td>0.2070</td>
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<td>2002 01-06</td>
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<td>0.3270</td>
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<td>0.2208</td>
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<td>0.1243</td>
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<tr>
<td>2004 07-12</td>
<td>0.1227</td>
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<td>0.1121</td>
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<tr>
<td>2005 01-06</td>
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<td>2008 01-06</td>
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<tr>
<td>2008 07-12</td>
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<td>0.3296</td>
<td>0.3435</td>
<td>0.3754</td>
<td>0.4017</td>
<td>0.4722</td>
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</tbody>
</table>

This table reports the implied volatilities calculated by inverting the Black and Scholes (1973) model separately for each moneyness category. The implied volatilities of individual options are then averaged within each moneyness category and across twenty-six consecutive six-month subperiods. Moneyness is defined as $S/K$, where $S$ denotes the spot price and $K$ denotes the strike price. 1996 01-06 denotes the period from January 1996 to June 1996.

- (1) $S/K < 0.94$.
- (2) $0.94 < S/K < 0.97$.
- (3) $0.97 < S/K < 1.00$.
- (4) $1.00 < S/K < 1.03$.
- (5) $1.03 < S/K < 1.06$.
- (6) $S/K > 1.06$.

deep OTM calls. That is, a volatility smile is skewed toward one side. The skewed volatility smile is sometimes called a volatility sneer because it looks more like a sardonic smile than a sincere smile. In the equity options market, the volatility sneer is often negatively skewed, where lower strike prices for OTM puts have higher implied volatilities, and thus higher valuations (Rubinstein 1994; Bakshi et al 1997). This result is consistent with those of Rubinstein (1994), Derman (1999), Bakshi et al (2001) and Dennis and Mayhew (2002). Since the smile evidence is indicative of a deep OTM calls. That is, a volatility smile is skewed toward one side. The skewed volatility smile is sometimes called a volatility sneer because it looks more like a sardonic smile than a sincere smile. In the equity options market, the volatility sneer is often negatively skewed, where lower strike prices for OTM puts have higher implied volatilities, and thus higher valuations (Rubinstein 1994; Bakshi et al 1997). This result is consistent with those of Rubinstein (1994), Derman (1999), Bakshi et al (2001) and Dennis and Mayhew (2002). Since the smile evidence is indicative of a
negatively skewed implicit return distribution with excess kurtosis, a better model must be based on a distributional assumption that allows for negative skewness and excess kurtosis.

4 EMPIRICAL RESULTS

In this section, we examine the empirical performances of each model with respect to in-sample pricing, out-of-sample pricing and hedging performance. The analysis is based on two measures, ie, mean absolute error (MAE) and root mean square error (RMSE), respectively:

\[
\text{MAE} = \frac{1}{T} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} |O_{i,t} - O_{i,t}^*|,
\]

\[
\text{RMSE} = \sqrt{\frac{1}{T} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} (O_{i,t} - O_{i,t}^*)^2},
\]

where, for option \(i\) on day \(t\), \(O_{i,t}^*\) denotes the model price, \(O_{i,t}\) denotes the market price, \(N\) denotes the number of options and \(T\) denotes the number of days in the sample. The MAE measures the magnitude of the pricing errors, while the RMSE measures the volatility of the errors. The option-pricing model with the lowest values for both the MAE and RMSE will be the best.

4.1 In-sample pricing performance

Table 3 reports the mean and standard error of the parameter estimates for each model. The \(R^2\) values for each AHBS-type model are also reported. For the AHBS-type models, each parameter is estimated by the OLS every day. For the BS, SV and SVJ models, each parameter is estimated by minimizing the sum of the squared errors between the model and the market option prices every day. First, the daily estimates of each model’s parameters have excessive standard errors. However, such an estimation is valuable for the following reasons. The estimated parameters can be generated by indicating market sentiment on a daily basis and can suggest future specifications of more complicated dynamic models. In addition, because the AHBS-type models are based not on theoretical backgrounds but on trader rules, excessive standard errors of daily estimates are not a fatal problem. Second, as expected, the R3 and A3 models, which have four independent variables, show higher \(R^2\) values than the other models. Therefore, it is necessary to check for the overfitting problem by examining out-of-sample pricing and hedging performance. Third, the implied correlation between the index return and the volatility levels of the SV and SVJ models has a negative
### TABLE 3  Parameters.

(a) AHBS-type models

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<tr>
<th></th>
<th>( \beta_0 )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( \beta_3 )</th>
<th>( R^2 )</th>
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<td>(0.1255)</td>
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<tr>
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<td>-77.5085</td>
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<tr>
<td></td>
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<td>(3.0511)</td>
<td>(2.9996)</td>
<td>(0.9828)</td>
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<tr>
<td>A1</td>
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<tr>
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<td>(0.0000)</td>
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<tr>
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<td>(0.0000)</td>
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<tr>
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<td>-0.0001</td>
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<td>(0.0030)</td>
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<td>(0.0000)</td>
<td>(0.0285)</td>
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(b) Other models

<table>
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<table>
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<tr>
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<th>( \mu_J )</th>
<th>( \sigma_J )</th>
<th>( \kappa_v )</th>
<th>( \theta_v )</th>
<th>( \sigma_v )</th>
<th>( \rho )</th>
<th>( v_t )</th>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
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<td>0.3722</td>
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<td>0.0390</td>
<td>(0.0638)</td>
<td>(0.0417)</td>
<td>(0.0264)</td>
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<td>(0.0046)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>SVJ</td>
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<td>0.2032</td>
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</tr>
<tr>
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<td>(0.0087)</td>
<td>(0.0172)</td>
<td>(0.0258)</td>
<td>(0.0212)</td>
<td>(0.0083)</td>
<td>(0.0201)</td>
<td>(0.0007)</td>
</tr>
</tbody>
</table>

This table reports the mean and the standard error of the parameter estimates for each model. The mean and the standard deviation of \( R^2 \) values for each AHBS-type model are reported. For the AHBS-type models, each parameter is estimated by ordinary least squares every day. For the BS, SV and SVJ models, each parameter is estimated by minimizing the sum of the squared errors between the model and the market option prices every day. R1 is the ad hoc BS model that considers the intercept and the moneyness as the independent variables. R2 is the ad hoc BS model that considers the intercept, the moneyness and the square of the moneyness. R3 is the ad hoc BS model that considers the intercept, the moneyness and the square and the third power of the moneyness. A1 is the ad hoc BS model that considers the intercept and the strike price. A2 is the ad hoc BS model that considers the intercept, the strike price and the square of the strike price. A3 is the ad hoc BS model that considers the intercept, the strike price and the square and the third power of the strike price. BS is the Black and Scholes (1973) option-pricing model. SV is the option-pricing model considering the continuous-time stochastic volatility. SVJ is the option-pricing model considering the continuous-time stochastic volatility and the jumps.

value. The negative estimate indicates that the implied volatility and index returns are negatively correlated, and the implied risk-neutral distribution recognized by option traders is negatively skewed. In addition, the mean of the jumps of the SVJ model is negative. This result is consistent with the volatility sneer pattern shown in Table 2.
### TABLE 4  In-sample pricing errors.

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>BS</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
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<tbody>
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<td>0.3100</td>
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<td>0.3979</td>
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<td>0.6187</td>
<td>0.3571</td>
<td>0.1891</td>
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<td>1.5185</td>
<td>0.9628</td>
<td>0.8138</td>
<td>0.5887</td>
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<td>(4)</td>
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<td>(5)</td>
<td>2.2481</td>
<td>0.3666</td>
<td>0.2984</td>
<td>0.2129</td>
<td>0.3854</td>
<td>0.2981</td>
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<td>0.3160</td>
<td>0.5588</td>
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<td>0.2927</td>
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</table>

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>BS</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
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</thead>
<tbody>
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<td>0.2383</td>
<td>2.8438</td>
<td>0.3796</td>
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<td>0.8813</td>
<td>0.9327</td>
<td>1.0172</td>
<td>0.9080</td>
<td>0.8887</td>
<td>0.7399</td>
<td>0.7221</td>
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<td>(5)</td>
<td>2.6427</td>
<td>0.5752</td>
<td>0.5852</td>
<td>0.3648</td>
<td>0.5569</td>
<td>0.5734</td>
<td>0.3191</td>
<td>0.2791</td>
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<td>(6)</td>
<td>2.5675</td>
<td>0.3225</td>
<td>0.3329</td>
<td>0.2059</td>
<td>0.3643</td>
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<td>0.1828</td>
<td>0.2681</td>
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<td><strong>Total</strong></td>
<td>2.3618</td>
<td>0.8635</td>
<td>0.7191</td>
<td>0.5752</td>
<td>1.2557</td>
<td>0.7065</td>
<td>0.5432</td>
<td>0.4731</td>
<td>0.6114</td>
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</table>

This table reports in-sample pricing errors with respect to moneyness. The in-sample pricing performance of each model is evaluated by comparing the market prices with the model's prices computed using the parameter estimates from the current day. $S/K$ is defined as moneyness, where $S$ denotes the asset price and $K$ denotes the strike price. MAE denotes mean absolute errors and RMSE denotes root mean square error. R1 is the ad hoc BS model that considers the intercept and the moneyness as the independent variables. R2 is the ad hoc BS model that considers the intercept, the moneyness and the square of the moneyness. R3 is the ad hoc BS model that considers the intercept, the moneyness, the square and the third power of the moneyness. A1 is the ad hoc BS model that considers the intercept and the strike price. A2 is the ad hoc BS model that considers the intercept, the strike price and the square of the strike price. A3 is the ad hoc BS model that considers the intercept, the strike price and the square and the third power of the strike price. BS is the Black and Scholes (1973) option-pricing model. SV is the option-pricing model considering the continuous-time stochastic volatility. SVJ is the option-pricing model considering the continuous-time stochastic volatility and jumps. (1) $S/K < 0.94$. (2) $0.94 < S/K < 0.97$. (3) $0.97 < S/K < 1.00$. (4) $1.00 < S/K < 1.03$. (5) $1.03 < S/K < 1.06$. (6) $S/K > 1.06$.

We evaluate the in-sample pricing performance of each model by comparing the market prices with the model’s prices computed with the parameter estimates from the current day. Table 4 reports the in-sample valuation errors for the alternative models computed over the whole sample of options. The SVJ model shows the best performance, closely followed by the SV model for the MAE; the SV model outperforms other models for the RMSE. Roughly, the (mathematically complicated) SV and SVJ models are better than the AHBS-type models for in-sample pricing. In-sample pricing
performance is simply contingent on the number of free parameters. It is obvious that
the option-pricing model that has the most free parameters gives the best performance.
In addition, all the models show moneyness-based valuation errors. They exhibit the
worst fit for the OTM options. The fit, as measured by the MAE, steadily improves as
we move from near-the-money to OTM options. In addition, the SV and SVJ mod-
els do not perform better than the AHBS-type models for OTM call options when
$S/K < 1$.

Overall, all AHBS-type models and mathematically complicated models perform
better than the BS model. In addition, trader rules can explain current market prices
in the options market, although this is not rooted in rigorous theory.

4.2 Out-of-sample pricing performance

In-sample pricing performance can be perverted due to the dormant problem of over-
fitting the data. A good in-sample fit could be the result of increasingly larger numbers
of parameters. To reduce the effect of this connection to inferences, we examine the
model’s out-of-sample pricing performance. In out-of-sample pricing, the presence
of more parameters could actually cause overfitting, and thus the model could be
penalized if the extra parameters do not improve its structural fitting. This analysis
also aims to evaluate the stability of each model’s parameters over time. To control
the parameters’ stability over different periods, we analyze out-of-sample valuation
errors for one day or one week. We use the current day’s estimated parameters to price
options for the following day (or week).

Tables 5 and 6, respectively, report one-day-ahead and one-week-ahead out-of-
sample pricing errors, computed over the whole sample of options, for the different
models. First, we examine out-of-sample pricing performance using the nearest-to-
next rollover strategy. Panel (a) of Tables 5 and 6 reports the results using the nearest-
to-next rollover strategy. For one-day-ahead out-of-sample pricing, the A2 model
shows the best performance, closely followed by the A1 model. For the one-week-
ahead out-of-sample pricing, the A1 model exhibits the better fit. The mathematically
complicated models are competitive for in-sample pricing performance. However,
for the out-of-sample pricing performance, the AHBS-type models perform better
than the SV and SVJ models. In addition, for in-sample pricing performance, the A3
and R3 models, which have more parameters than the other models, perform better
than simpler models. However, for out-of-sample pricing performance, the simpler
A1 and A2 models are better. That is, the presence of more parameters actually causes
and Kim (2009), the trader rules dominate more mathematically sophisticated models,
although the SV and SVJ models are not far behind. With respect to moneyness-based
errors, similar to the results of in-sample pricing performance, the MAE steadily
### Table 5
One-day-ahead out-of-sample pricing errors. [Table continues on next page.]

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>BS</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S/K &lt; 0.94 )</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0.94 &lt; S/K &lt; 0.96 )</td>
<td>2.3820</td>
<td>1.7209</td>
<td>1.4492</td>
<td>1.7704</td>
<td>1.7932</td>
<td>1.2515</td>
<td>1.7001</td>
<td>1.0524</td>
<td>1.3395</td>
</tr>
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<td>( 0.96 &lt; S/K &lt; 1.00 )</td>
<td>2.0893</td>
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<td>1.0759</td>
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<td>0.7952</td>
<td>1.0933</td>
<td>1.2787</td>
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<td>( 1.00 &lt; S/K &lt; 1.03 )</td>
<td>1.9595</td>
<td>1.6458</td>
<td>1.5069</td>
<td>1.3948</td>
<td>1.4724</td>
<td>1.2002</td>
<td>1.0755</td>
<td>1.5138</td>
<td>1.6761</td>
</tr>
<tr>
<td>( 1.03 &lt; S/K &lt; 1.06 )</td>
<td>1.8610</td>
<td>1.3506</td>
<td>1.2755</td>
<td>1.3002</td>
<td>1.1108</td>
<td>1.0091</td>
<td>1.0219</td>
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<td>1.5424</td>
</tr>
<tr>
<td>( S/K &gt; 1.06 )</td>
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<td>0.9832</td>
<td>1.5465</td>
<td>0.5317</td>
<td>0.7510</td>
<td>1.1171</td>
<td>0.6946</td>
<td>0.6744</td>
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<tr>
<td>Total</td>
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<td>1.1536</td>
<td>1.1653</td>
<td>1.3370</td>
<td>1.0080</td>
<td>0.9187</td>
<td>1.0406</td>
<td>1.1001</td>
<td>1.1923</td>
</tr>
</tbody>
</table>

| \( S/K < 0.94 \) | 4.3852 | 9.8866 | 6.9483 | 11.2266 | 12.1547 | 7.4598 | 11.6676 | 2.4316 | 3.6415 |
| \( 0.94 < S/K < 0.96 \) | 3.0969 | 2.5736 | 2.2606 | 2.2050 | 2.8509 | 1.9258 | 1.8610 | 2.3508 | 3.3254 |
| \( 0.96 < S/K < 1.00 \) | 2.7602 | 2.6426 | 2.3948 | 2.3005 | 2.3494 | 1.9160 | 1.7958 | 2.8194 | 3.5162 |
| \( 1.00 < S/K < 1.03 \) | 2.7072 | 2.2118 | 2.1536 | 2.1869 | 1.8059 | 1.7111 | 1.7057 | 2.7001 | 3.1183 |
| \( 1.03 < S/K < 1.06 \) | 3.0361 | 1.7302 | 1.7318 | 1.8537 | 1.3526 | 1.3127 | 1.2769 | 2.1421 | 2.5299 |
| \( S/K > 1.06 \) | 2.8041 | 1.6942 | 3.1487 | 12.5589 | 1.3521 | 2.3913 | 9.4232 | 1.6386 | 2.1283 |
TABLE 5  Continued.

(b) Next-to-next

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>BS</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
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<td>$S/K &lt; 0.94$</td>
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<td>1.0851</td>
<td>0.7968</td>
<td>1.0984</td>
<td>0.9585</td>
<td>0.9453</td>
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<td>1.0664</td>
<td>0.9990</td>
<td>0.9275</td>
<td>0.8735</td>
<td>0.7367</td>
<td>0.6670</td>
<td>0.9400</td>
<td>0.9276</td>
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<td>1.3561</td>
<td>1.1138</td>
<td>0.9807</td>
<td>1.3139</td>
<td>1.3267</td>
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<td>1.2706</td>
<td>1.1821</td>
<td>1.2049</td>
<td>1.0167</td>
<td>0.9067</td>
<td>0.9146</td>
<td>1.2125</td>
<td>1.2527</td>
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<td>$1.03 &lt; S/K &lt; 1.06$</td>
<td>2.2271</td>
<td>0.8463</td>
<td>0.8161</td>
<td>0.8077</td>
<td>0.6525</td>
<td>0.6072</td>
<td>0.5768</td>
<td>0.8584</td>
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<td>$S/K &gt; 1.06$</td>
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<td>0.5325</td>
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<td>0.4249</td>
<td>0.3838</td>
<td>0.3861</td>
<td>0.5694</td>
<td>0.5258</td>
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<td><strong>Total</strong></td>
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<td>1.0144</td>
<td>0.9502</td>
<td>0.9467</td>
<td>0.8399</td>
<td>0.7158</td>
<td>0.7095</td>
<td>0.9335</td>
<td>0.9284</td>
</tr>
</tbody>
</table>

| $0.94 < S/K < 0.96$ | 2.9122 | 1.9145 | 1.8257 | 1.7539 | 1.5357 | 1.3176 | 1.2002 | 1.7983 | 1.8402 |
| $0.96 < S/K < 1.00$ | 2.6059 | 2.3118 | 2.1726 | 2.0654 | 1.9846 | 1.6376 | 1.4830 | 2.1155 | 2.1516 |
| $1.00 < S/K < 1.03$ | 2.4388 | 1.9392 | 1.8888 | 1.9098 | 1.4814 | 1.3808 | 1.3534 | 1.8983 | 1.9627 |
| $1.03 < S/K < 1.06$ | 2.8444 | 1.4731 | 1.4869 | 1.4545 | 1.0535 | 1.0306 | 0.9614 | 1.4924 | 1.5424 |
| $S/K > 1.06$ | 2.6668 | 1.1429 | 1.1481 | 1.1399 | 0.8229 | 0.7548 | 0.7765 | 1.1794 | 1.1557 |
| **Total** | 2.7665 | 2.5918 | 2.1065 | 3.4315 | 2.7239 | 1.7325 | 3.2841 | 1.7102 | 1.7372 |

This table reports one-day-ahead out-of-sample pricing errors with respect to moneyness. Each model is estimated every day during the sample period; one-day-ahead out-of-sample pricing errors are computed using the estimated parameters from the previous trading day. Panel (a) reports one-day-ahead out-of-sample pricing errors using the nearest-to-next rollover strategy. Panel (b) reports one-day-ahead out-of-sample pricing errors using the next-to-next rollover strategy. $S/K$ is defined as moneyness, where $S$ denotes the asset price and $K$ denotes the strike price. MAE denotes mean absolute error and RMSE denotes root mean square error. R1 is the ad hoc BS model that considers the intercept and the moneyness as the independent variables. R2 is the ad hoc BS model that considers the intercept, the moneyness and the square of the moneyness. R3 is the ad hoc BS model that considers the intercept, the moneyness, the square and the third power of the moneyness. A1 is the ad hoc BS model that considers the intercept and the strike price. A2 is the ad hoc BS model that considers the intercept, the strike price and the square of the strike price. A3 is the ad hoc BS model that considers the intercept, the strike price and the square and the third power of the strike price. BS is the Black and Scholes (1973) option-pricing model. SV is the option-pricing model considering the continuous-time stochastic volatility. SVJ is the option-pricing model considering the continuous-time stochastic volatility and jumps.
### TABLE 6  One-week-ahead out-of-sample pricing errors. [Table continues on next page.]

<table>
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<tr>
<th>Moneyness</th>
<th>BS</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MAE</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S/K &lt; 0.94 )</td>
<td>2.6955</td>
<td>2.4740</td>
<td>2.5531</td>
<td>3.0536</td>
<td>2.4607</td>
<td>2.3104</td>
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<td>1.9402</td>
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<td>2.3209</td>
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<td>1.6174</td>
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<td>1.3106</td>
<td>1.3781</td>
<td>2.0172</td>
<td>2.6683</td>
</tr>
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<td>2.4645</td>
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<td>2.8080</td>
<td>3.0853</td>
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<tr>
<td>( 1.03 &lt; S/K &lt; 1.06 )</td>
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<td>1.5557</td>
<td>1.5498</td>
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<td>1.1563</td>
<td>1.1495</td>
<td>1.1733</td>
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<td>2.2210</td>
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<tr>
<td>( S/K &gt; 1.06 )</td>
<td>1.9914</td>
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<td>1.5101</td>
<td>1.5898</td>
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<td>2.3977</td>
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<td><strong>RMSE</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S/K &lt; 0.94 )</td>
<td>5.4083</td>
<td>11.7077</td>
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<td>14.5579</td>
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<tr>
<td>( 0.94 &lt; S/K &lt; 0.96 )</td>
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<td>3.6598</td>
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<tr>
<td>( S/K &gt; 1.06 )</td>
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<td>3.4903</td>
<td>11.2947</td>
<td>3.0310</td>
<td>3.0275</td>
</tr>
</tbody>
</table>
This table reports one-week-ahead out-of-sample pricing with respect to moneyness. Each model is estimated every day during the sample period; one-week-ahead out-of-sample pricing errors are computed using estimated parameters from one week ago. Panel (a) reports one-week-ahead out-of-sample pricing errors using the nearest-to-next rollover strategy. Panel (b) reports one-week-ahead out-of-sample pricing errors using the next-to-next rollover strategy. $S/K$ is defined as moneyness, where $S$ denotes the asset price and $K$ denotes the strike price. MAE denotes mean absolute errors and RMSE denotes root mean square error. R1 is the ad hoc BS model that considers the intercept and the moneyness as the independent variables. R2 is the ad hoc BS model that considers the intercept, the moneyness and the square of the moneyness. R3 is the ad hoc BS model that considers the intercept, the moneyness, the square and the third power of the moneyness. A1 is the ad hoc BS model that considers the intercept and the strike price. A2 is the ad hoc BS model that considers the intercept, the strike price and the square of the strike price. A3 is the ad hoc BS model that considers the intercept, the strike price and the square and the third power of the strike price. BS is the Black and Scholes (1973) option-pricing model. SV is the option-pricing model considering the continuous-time stochastic volatility. SVJ is the option-pricing model considering the continuous-time stochastic volatility and jumps.

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>BS</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
</tr>
</thead>
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<tr>
<td>$S/K &lt; 0.94$</td>
<td>2.3866</td>
<td>1.6770</td>
<td>1.7050</td>
<td>1.5188</td>
<td>1.2440</td>
<td>1.2787</td>
<td>1.0834</td>
<td>1.5221</td>
<td>1.3450</td>
</tr>
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<td>1.5110</td>
<td>1.4381</td>
<td>1.1851</td>
<td>1.0791</td>
<td>0.9971</td>
<td>1.5652</td>
<td>1.4809</td>
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<td>1.7648</td>
<td>1.541</td>
<td>1.4046</td>
<td>2.2169</td>
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<td>1.9167</td>
<td>1.9159</td>
<td>1.4130</td>
<td>1.2908</td>
<td>1.2917</td>
<td>2.1895</td>
<td>2.1238</td>
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decreases as we move from near-the-money to deep OTM options for all the models. Therefore, the simple “absolute smile” AHBS approaches – the A1 and A2 models – outperform all the other models.

The size of the pricing errors increases as we move from in-sample to out-of-sample pricing. The average MAE of all the models is 0.5360 for in-sample pricing; this increases to 1.2184 for one-day-ahead out-of-sample pricing. One-week-ahead out-of-sample pricing errors grow to 1.9805, which is almost four times as much as the in-sample pricing errors. The relative margin of pricing performance changes significantly, unlike that of the in-sample pricing results. The difference between the BS model and the best model decreases in out-of-sample pricing. The ratio of the BS model to the SVJ model for the MAE is 8.0882 for in-sample pricing errors. The ratio of the BS model to the A2 or A1 model decreases to 2.2312 and to 1.5415 for one-day-ahead and one-week-ahead out-of-sample errors, respectively. As the out-of-sample pricing period increases, the difference between the BS model and the best model decreases. The pricing performance of the SVJ model, which is the best model for in-sample pricing, is not maintained as the out-of-sample pricing period increases, implying that the presence of more parameters actually causes overfitting. The best of the AHBS-type models, A3 and R3, for in-sample pricing do not hold their position for one-day-ahead and one-week-ahead out-of-sample pricing: for out-of-sample pricing, the A3 and R3 models perform worst, implying that the presence of more parameters actually causes overfitting. This result is consistent with the results of Jackwerth and Rubinstein (2012), Li and Pearson (2007) and Kim (2009). Therefore, both the mathematically complicated option-pricing models and the AHBS-type models have overfitting problems when the traditional nearest-to-next rollover strategy is used. To mitigate these problems, we need to consider the new rollover strategy: the next-to-next strategy.

Next, we examine pricing performance using the next-to-next rollover strategy, as suggested by Choi and Ok (2012). Panel (b) of Tables 5 and 6 reports the results using the next-to-next rollover strategy. Above all, the next-to-next rollover strategy decreases the pricing errors of all the option-pricing models. After using the next-to-next strategy, we find the averages of the MAEs of all the option-pricing models decrease by 20% (35%), from 1.2184 (1.9805) to 1.0046 (1.4687), for one-day-ahead (one-week-ahead) out-of-sample pricing. Parts (a) and (b) of Figure 2 illustrate the MAE of each option-pricing model for the nearest-to-next and next-to-next rollover strategies, respectively. The pricing errors of all the models are greatly decreased by the next-to-next strategy. The models with more parameters (R3, A3 and SVJ) perform best. For both one-day-ahead and one-week-ahead out-of-sample pricing, A3 generally gives the best performance. When the nearest-to-next rollover strategy is applied, the simpler models, A1 and A2, perform better than the others. However, using the next-to-next strategy, we find A3 performs better than the complicated
models. Therefore, when the next-to-next rollover strategy is applied, the overfitting problems disappear and A3 outperforms all the other models.

Finally, we examine the relative strength of the absolute and relative smile approaches for pricing options. For in-sample pricing performance, the average MAEs of the “relative smile” and “absolute smile” approaches are 0.4092 and 0.4129, respectively. Using the nearest-to-next strategy, for one-day-ahead (one-week-ahead) out-of-sample pricing, we find the average MAEs of alternative “relative smile” and “absolute smile” approaches are 1.2186 (2.0395) and 0.9891 (1.6234), respectively.
Using the next-to-next strategy, for one-day-ahead (one-week-ahead) out-of-sample pricing, we find the averages of the MAEs of the alternative “relative smile” and “absolute smile” approaches are 0.9704 (1.5087) and 0.7551 (1.0833), respectively. Irrespective of the type of rollover strategy, the reduction in pricing errors for the absolute smile approach has less effect than the relative smile approach. This result is consistent with those of Jackwerth and Rubinstein (2012), Li and Pearson (2007), Kim (2009) and Choi and Ok (2012), who report that the “absolute smile” model beats the “relative smile” model in predicting prices. This result can be explained by the fact that the absolute smile model implicitly adjusts for the negative correlation between the index return and the volatility level. Because the absolute model treats skewness as a fixed function of the strike price, \( S/K \), instead of moneyness, it leads to lower implied volatility than the relative smile model when there is an increase in the stock price.

4.3 Hedging performance

Hedging performance is an important tool for gauging the forecasting power of the volatility of underlying assets. In practice, option traders usually focus on risk due to the underlying asset price volatility alone and carry out a delta-neutral hedge, employing only the underlying asset as the hedging instrument. While this seems plausible for the BS and AHBS-type models, the SV and SVJ models lead to incomplete markets. It is well known that a simple delta-hedging strategy is suboptimal in this setting. In addition, Alexander and Nogueira (2007) show that the delta hedge ratios of the SV and SVJ models should theoretically be identical (or only be driven by differences in the model fit) because of the homogeneity of the call option prices and the scale invariance of the SV and SVJ models.

Because there are several risk factors in the proposed SV and SVJ models, the need for a perfect hedge could arise in situations where not only the underlying price risk, but also the volatility, or jump risk, is present. To implement this hedging practice, we should recognize that a perfect hedge is not practically feasible in the presence of stochastic jumps. Thus, in line with the measure of hedging performances of Dumas et al. (1998) and Gemmill and Sailekos (2000), we define the hedging error as

\[
\varepsilon_t = \Delta O - \Delta O^*,
\]

where \( \Delta O \) is the change in the reported market price from day \( t \) until day \( t + 1 \) or \( t + 7 \), and \( \Delta O^* \) is the change in the model’s theoretical price.

Tables 7 and 8, respectively, present one-day and one-week hedging errors over alternative moneyness categories. First, using the nearest-to-next rollover strategy, we find the A3 model has the best hedging performance for one day and one week. The SV
This figure shows the mean absolute errors (MAE) of hedging for each option-pricing model with respect to the rollover strategies. Panel (a) represents one-day-ahead hedging errors and panel (b) represents one-week-ahead hedging errors. R1 is the ad hoc BS model that considers the intercept and the moneyness as the independent variables. R2 is the ad hoc BS model that considers the intercept, the moneyness and the square of the moneyness. R3 is the ad hoc BS model that considers the intercept, the moneyness, the square and the third power of the moneyness. A1 is the ad hoc BS model that considers the intercept and the strike price. A2 is the ad hoc BS model that considers the intercept, the strike price and the square of the strike price. A3 is the ad hoc BS model that considers the intercept, the strike price and the square and the third power of the strike price. BS is the Black and Scholes (1973) option-pricing model. SV is the option-pricing model considering the continuous-time stochastic volatility. SVJ is the option-pricing model considering the continuous-time stochastic volatility and the jumps.

and SVJ models are the worst performers: worse even than the BS model. The AHBS-type models show better hedging performance than the other models. The ratios of the BS model to the A3 model (which is the best performer) are 1.2915 and 1.3866 for one-day-ahead and one-week-ahead hedging errors, respectively. As the hedging period increases, the difference between the BS model and the best model decreases. Second, we examine hedging performances using the next-to-next rollover strategy. In Figure 3, the hedging errors of all the models are decreased and the complicated models are favored the most, similar to the out-of-sample pricing results. The A3
### TABLE 7  One-day-ahead hedging errors. [Table continues on next page.]

(a) Nearest-to-next

<table>
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<tr>
<th>Moneyness</th>
<th>BS</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
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<tbody>
<tr>
<td>$S/K &lt; 0.94$</td>
<td>1.4032</td>
<td>1.1922</td>
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<td>0.9999</td>
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<td>1.4638</td>
<td>1.5533</td>
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<td>1.2066</td>
<td>1.2221</td>
<td>1.2702</td>
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<td>0.6573</td>
<td>0.6568</td>
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<td>0.4380</td>
<td>0.4349</td>
<td>0.5749</td>
<td>0.5980</td>
</tr>
</tbody>
</table>

| Total | 0.8441 | 0.8649 | 0.8584 | 0.8466 | 0.6664 | 0.6608 | 0.6536 | 0.9048 | 0.9385 |

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<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
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<td>1.5585</td>
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<td>2.0838</td>
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| Total | 1.6723 | 1.8186 | 1.7986 | 1.7842 | 1.4585 | 1.4362 | 1.4274 | 2.0078 | 2.4716 |
Table 7 Continued.

(b) Next-to-next

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<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>SV</th>
<th>SVJ</th>
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This table reports one-day-ahead hedging error with respect to moneyness. For each option, hedging error is the difference between the change in the reported market price and the change in the model's theoretical price from day $t$ until day $t+1$. Panel (a) reports one-day-ahead hedging errors using the nearest-to-next rollover strategy. Panel (b) reports one-day-ahead hedging errors using the next-to-next rollover strategy. MAE denotes mean absolute errors and RMSE denotes root mean square error. R1 is the ad hoc BS model that considers the intercept and the moneyness as the independent variables. R2 is the ad hoc BS model that considers the intercept, the moneyness and the square of the moneyness. R3 is the ad hoc BS model that considers the intercept, the moneyness, the square and the third power of the moneyness. A1 is the ad hoc BS model that considers the intercept and the strike price. A2 is the ad hoc BS model that considers the intercept, the strike price and the square of the strike price. A3 is the ad hoc BS model that considers the intercept, the strike price and the square and the third power of the strike price. BS is the Black and Scholes (1973) option-pricing model. SV is the option-pricing model considering the continuous-time stochastic volatility. SVJ is the option-pricing model considering the continuous-time stochastic volatility and jumps.
<table>
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<th>Moneyness</th>
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<th>R2</th>
<th>R3</th>
<th>A1</th>
<th>A2</th>
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<td>S/K &lt; 0.94</td>
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| **RMSE** |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $S/K < 0.94$ | 3.1783 | 2.9651 | 2.8073 | 2.6715 | 3.7514 | 2.0978 | 1.9467 | 2.9383 | 2.7330 |
| $0.94 < S/K < 0.96$ | 2.3139 | 2.5200 | 2.4310 | 2.3587 | 1.8485 | 1.7591 | 1.6915 | 2.6926 | 2.4580 |
| $0.96 < S/K < 1.00$ | 2.2742 | 2.8370 | 2.7691 | 2.7306 | 2.0650 | 1.9893 | 1.9214 | 3.3470 | 2.8145 |
| $1.00 < S/K < 1.03$ | 2.5007 | 2.7508 | 2.7457 | 2.8430 | 1.9779 | 1.9697 | 1.9596 | 2.8990 | 3.1287 |
| $1.03 < S/K < 1.06$ | 2.2279 | 2.1269 | 2.1603 | 2.2536 | 1.5544 | 1.5348 | 1.5202 | 2.4300 | 2.3431 |
| $S/K > 1.06$ | 2.1324 | 1.6784 | 1.7441 | 1.6765 | 1.2180 | 1.2029 | 1.1879 | 1.9382 | 1.8766 |

Total: 1.1352 1.0446 1.0278 1.0138 0.7799 0.7463 0.7305 1.1277 1.0730

Total: 2.0121 2.0287 2.0101 2.0006 1.6541 1.4402 1.4024 2.2453 2.1268

This table reports one-week-ahead hedging error with respect to moneyness. For each option, hedging error is the difference between the change in the reported market price and the change in the model’s theoretical price from day $t$ until day $t + 7$. Panel (a) reports one-week-ahead hedging errors using the nearest-to-next rollover strategy. Panel (b) reports one-week-ahead hedging errors using the next-to-next rollover strategy. MAE denotes mean absolute errors and RMSE denotes root mean square error. R1 is the ad hoc BS model that considers the intercept and the moneyness as the independent variables. R2 is the ad hoc BS model that considers the intercept, the moneyness and the square of the moneyness. R3 is the ad hoc BS model that considers the intercept, the moneyness, the square and the third power of the moneyness. A1 is the ad hoc BS model that considers the intercept and the strike price. A2 is the ad hoc BS model that considers the intercept, the strike price and the square of the strike price. A3 is the ad hoc BS model that considers the intercept, the strike price and the square and the third power of the strike price. BS is the Black and Scholes (1973) option-pricing model. SV is the option-pricing model considering the continuous-time stochastic volatility. SVJ is the option-pricing model considering the continuous-time stochastic volatility and jumps.
model is the best performer for both one-day-ahead and one-week-ahead hedging errors. After the next-to-next approach is applied to the models, the SV and SVJ models outperform the BS model. As with the out-of-sample pricing results, the next-to-next strategy can mitigate the overfitting problem of AHBS-type models. The AHBS-type models that have more parameters show better hedging performance than those with fewer parameters. However, the next-to-next strategy does not drastically decrease the hedging errors. The next-to-next rollover strategy can decrease one-day-ahead and one-week-ahead out-of-sample pricing errors by 18% and 26%, respectively, but can decrease one-day-ahead and one-week-ahead hedging errors by only 9% and 16%, respectively. Therefore, the next-to-next rollover strategy can also decrease hedging errors, but not drastically.

4.4 Robustness check

Unlike what we did using the two measures MAE and RMSE, we now compare the performance of several option-pricing models by using a statistical test to arrive at concrete results. Table 9 reports the pairwise comparison results of the models by providing the $t$-statistics of the probability that the errors of one model are larger than those of the other. Parts (a) and (b), respectively, report the $t$-statistics between the one-day-ahead out-of-sample pricing errors for each model using the nearest-to-next and next-to-next rollover strategies. Parts (c) and (d), respectively, report the $t$-statistics between the one-week-ahead out-of-sample pricing errors for each model using the nearest-to-next and next-to-next rollover strategies. Parts (e) and (f), respectively, report the $t$-statistics between one-day-ahead hedging errors for each model using the nearest-to-next and next-to-next rollover strategies. Parts (g) and (h), respectively, report the $t$-statistics between the one-week-ahead hedging errors for each model using the nearest-to-next and next-to-next rollover strategies.

The comparison results are very clear and similar to those using the MAE. Almost all the differences between the errors for each model are statistically significant. When we use the traditional rollover method (the nearest-to-next strategy) for out-of-sample pricing, the A1 and A2 models perform better than the SV and SVJ models. In addition, the absolute smile approaches perform better than the relative smile approaches. As the out-of-sample pricing period increases, the A1 model increasingly gives superior results. For hedging performance, the absolute smile approaches perform better than the relative smile approaches and mathematically complicated models. However, the differences between absolute smile approaches are not significant. When we use the new rollover method, the next-to-next strategy, the A3 model is the best. The AHBS-type models (R3 and A3) with more parameters perform better than those with fewer parameters. That is, the next-to-next strategy can mitigate the overfitting problem of
the AHBS-type models. The differences in hedging performance of the nearest-to-next strategy and those for the next-to-next strategy are not so great.

Although we use sufficiently long sample periods, the results can be distorted by specific subperiods. Table 10 reports the MAEs of each model by subperiod. Parts (a) and (b) report the one-day-ahead and one-week-ahead out-of-sample pricing errors for each model, respectively. Parts (c) and (d) report the one-day-ahead and one-week-ahead hedging errors for each model, respectively. Bold numbers represent the smallest errors in each subperiod using the nearest-to-next-rollover–next-to-next-rollover strategy. The results for the subperiods are consistent with those for the full sample. The next-to-next strategy reduces the errors of all the option-pricing models. When the nearest-to-next rollover strategy is used, one of the “absolute smile” approaches is the best for pricing and hedging options, with a few exceptions. When we use the next-to-next strategy, model A3 becomes the unchallenged best performer, with a few exceptions. The AHBS-type models with more parameters perform better than those that have fewer parameters for pricing and hedging options. That is, the next-to-next strategy can resolve the overfitting problem of AHBS-type models.

5 CONCLUSION AND DISCUSSION

For S&P 500 options, we implemented a “horse race” competition between several option-pricing models. We considered trader rules to predict future implied volatilities by applying simple ad hoc rules to the observed current implied volatility patterns, as well as mathematically complicated option-pricing models, such as the stochastic volatility (and jumps) model, for pricing and hedging options. The rollover strategies for the parameters for each option-pricing model were also compared. In the nearest-to-next strategy, the option data for the nearest term contract on day \( t - 1 \) (or \( t - 7 \)) was used to estimate the parameters of the next-to-nearest contract on day \( t \), whereas in the next-to-next rollover strategy the next-to-nearest contract on day \( t - 1 \) (or \( t - 7 \)) was used to estimate the parameter of the next-to-nearest contract on day \( t \).

When we used the traditional rollover method, ie, the nearest-to-next strategy, we found that the stochastic volatility and jumps model was the best model for in-sample pricing. However, for out-of-sample pricing, the simple “absolute smile” trader rule, which assumes that the implied volatility follows a fixed function of the strike price, performed better than the mathematically complicated model. In addition, the “absolute smile” approaches performed better than the “relative smile” approaches, which assume that the implied volatility follows a fixed function of moneyness. Among simple trader rules, a simpler model with fewer parameters performed better than other models. That is, the presence of more parameters actually causes overfitting. The simple trader rules showed better hedging performance than the mathematically complicated models.
### TABLE 9  Differences between the errors for each model. [Table continues on next three pages.]

(a) One-day-ahead out-of-sample pricing using nearest-to-next

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<th>R1</th>
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<th>R3</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
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<th>SVJ</th>
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(b) One-day-ahead out-of-sample pricing using next-to-next

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This table reports the $t$-statistics of the difference between each model's absolute errors for pricing and hedging. Panel (a) reports $t$-statistics between one-day-ahead out-of-sample pricing errors for each model using the nearest-to-next rollover strategy. Panel (b) reports $t$-statistics between one-day-ahead out-of-sample pricing errors for each model using the next-to-next rollover strategy.
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Panel (c) reports t-statistics between one-week-ahead out-of-sample pricing errors for each model using the nearest-to-next rollover strategy. Panel (d) reports t-statistics between one-week-ahead out-of-sample pricing errors for each model using the next-to-next rollover strategy.
TABLE 9 Continued.

(e) One-day-ahead hedging using nearest-to-next

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(f) One-day-ahead hedging using next-to-next

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Panel (e) reports t-statistics between one-day-ahead hedging errors for each model using the nearest-to-next rollover strategy. Panel (f) reports t-statistics between one-day-ahead hedging errors for each model using the next-to-next rollover strategy.
Panel (g) reports \( t \)-statistics between one-week-ahead hedging errors for each model using the nearest-to-next rollover strategy. Panel (h) reports \( t \)-statistics between one-week-ahead hedging errors for each model using the next-to-next rollover strategy. Models R1, R2, R3, A2, A3, BS, SV and SVJ are defined in the text.
This table reports the mean absolute errors for each model for subperiods. Panel (a) reports one-day-ahead out-of-sample pricing errors for each model.

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This table reports the mean absolute errors for each model for subperiods. Panel (a) reports one-day-ahead out-of-sample pricing errors for each model.
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<td>1999</td>
<td>0.9010</td>
<td>0.8960</td>
<td>0.9030</td>
<td>0.8923</td>
<td><strong>0.7014</strong></td>
<td>0.7062</td>
<td>0.7032</td>
<td>1.0415</td>
<td>0.8864</td>
<td>0.8336</td>
<td>0.8311</td>
<td>0.8316</td>
<td>0.8220</td>
<td>0.6240</td>
<td>0.6271</td>
<td><strong>0.6237</strong></td>
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<td>0.8256</td>
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<tr>
<td>2000</td>
<td>0.7800</td>
<td>0.7885</td>
<td>0.7871</td>
<td>0.7837</td>
<td>0.6197</td>
<td>0.6195</td>
<td><strong>0.6189</strong></td>
<td>0.7716</td>
<td>0.7659</td>
<td>0.7065</td>
<td>0.7016</td>
<td>0.6999</td>
<td>0.6971</td>
<td>0.5428</td>
<td>0.5427</td>
<td><strong>0.5401</strong></td>
<td>0.6986</td>
<td>0.6840</td>
</tr>
</tbody>
</table>

Panel (c) reports one-day-ahead hedging errors for each model.
<table>
<thead>
<tr>
<th>Time Period</th>
<th>Nearest-to-next Rollover</th>
<th>Next-to-next Rollover</th>
</tr>
</thead>
<tbody>
<tr>
<td>1996 07-12</td>
<td>0.5775</td>
<td>0.4951</td>
</tr>
<tr>
<td>1997 07-12</td>
<td>0.5813</td>
<td>0.4588</td>
</tr>
<tr>
<td>1998 07-12</td>
<td>0.7080</td>
<td>0.5421</td>
</tr>
<tr>
<td>1999 07-12</td>
<td>0.6641</td>
<td>0.4968</td>
</tr>
<tr>
<td>2000 07-12</td>
<td>0.6610</td>
<td>0.4915</td>
</tr>
<tr>
<td>2001 07-12</td>
<td>0.6610</td>
<td>0.4915</td>
</tr>
<tr>
<td>2002 07-12</td>
<td>0.6610</td>
<td>0.4915</td>
</tr>
<tr>
<td>2003 07-12</td>
<td>0.6610</td>
<td>0.4915</td>
</tr>
<tr>
<td>2004 07-12</td>
<td>0.6610</td>
<td>0.4915</td>
</tr>
<tr>
<td>2005 07-12</td>
<td>0.6610</td>
<td>0.4915</td>
</tr>
<tr>
<td>2006 07-12</td>
<td>0.6610</td>
<td>0.4915</td>
</tr>
</tbody>
</table>

Panel (d) reports one-week-ahead hedging errors for each model. Bold numbers represent the smallest errors for each subperiod using the nearest-to-next rollover strategy.
When we used the new rollover method, the next-to-next strategy decreased the pricing and hedging errors of all the option-pricing models. The pricing errors of the simple trader rules were greatly decreased by the next-to-next strategy. Moreover, simple trader rules with more parameters performed better than those with fewer parameters for pricing options. That is, the next-to-next strategy can mitigate the overfitting problem of AHBS-type models. For hedging performance, the next-to-next strategy also decreased the errors of all the option-pricing models; however, the differences in the results using the nearest-to-next strategy and those using the next-to-next strategy were not so great.

It is obvious that the next-to-next rollover strategy performs better than the nearest-to-next strategy. Options prices include information on the distribution of the underlying asset at the option maturity date. After eliminating the nearest option contracts that expire in under seven days, we need to obtain information on the distribution of the underlying asset on the maturity date of the next-to-nearest contract for pricing and hedging the next-to-nearest options contracts with a time to maturity of over one month. In the nearest-to-next strategy, we use parameters that include information on the nearest option contracts to price and hedge the next-to-nearest options contracts. The maturity of the options that are priced is not consistent with that of the options from which the parameters are estimated. However, in the next-to-next rollover strategy we use parameters that include information on the next-to-nearest option contracts. Consistency between the maturity of options that are priced and that of the options from which the parameters are estimated is guaranteed. This can lead to better performance by the next-to-next rollover strategy than the nearest-to-next rollover strategy. In addition, if there are large differences in the performances of two rollover strategies, we can conjecture that the parameters estimated from the cross-sectional option data are totally different for each option maturity date.

Therefore, when the next-to-next strategy is considered, the “absolute smile” trader rule, which has more independent parameters, can be a competitive model for pricing and hedging S&P 500 index options.

DECLARATION OF INTEREST

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Research Paper

On empirical likelihood option pricing

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ABSTRACT

The Black–Scholes model is the golden standard for pricing derivatives and options in the modern financial industry. However, this method imposes some parametric assumptions on the stochastic process, and its performance becomes doubtful when these assumptions are violated. This paper investigates the application of a nonparametric method, namely the empirical likelihood (EL) method, in the study of option pricing. A blockwise EL procedure is proposed to deal with dependence in the data. Simulation and real data studies show that this new method performs reasonably well and, more importantly, outperforms classical models developed to account for jumps and stochastic volatility, thanks to the fact that nonparametric methods capture information about higher-order moments.

Keywords: nonparametric; option pricing; empirical likelihood; robust; blocking time series.

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1 INTRODUCTION

Since the seminal works of Black and Scholes (1973) and Merton (1973), option-valuation methodologies have developed extensively. The Black–Scholes model has become one of the most well-known discoveries in the finance literature, relating the cross-sectional properties of option prices with the underlying assets’ returns distributions. However, Rubinstein (1985) and Melino and Turnbull (1990) point out several limitations in the Black–Scholes model due to strong assumptions, such as the nonnormality of returns, stochastic volatility (implied volatility smile), jumps and others. Both parametric and nonparametric approaches have been proposed to deal with these issues.

Scott (1987), Hull and White (1987) and Wiggins (1987) extend the Black–Scholes model and allow the volatility to be stochastic. Heston (1993) develops a closed-form solution for option pricing when the underlying asset’s volatility is stochastic. Duan (1995) proposes a generalized autoregressive conditional heteroscedasticity (GARCH) option pricing model in an attempt to explain some systematic biases associated with the Black–Scholes model. Heston and Nandi (2000) provide a closed-form solution for option pricing, with the underlying asset’s volatility following a GARCH(\(p, q\)) process. Bates (1996) and Bakshi et al (1997) derive an option pricing model with stochastic volatility and jumps. Kou (2002) provides a solution to pricing the option with the double exponential jumps diffusion process. Carr and Madan (1999) introduce the fast Fourier transform approach to option pricing, given a specified characteristic function of the return, which provides an efficient computational algorithm to calculate the option prices (for more information, see, for example, Duffie et al (2000), Bakshi and Madan (2000) and Carr and Madan (2009)). All of these methods assume a parametric form of either the distribution of the underlying asset returns or the characteristic function of the underlying asset returns.

Nonparametric approaches have also been proposed to capture the underlying asset and option price data in order to reconstruct the structure of the diffusion process. For example, Hutchinson et al (1994) apply neural network techniques to price derivatives. Ait-Sahalia and Lo (1998) use the kernel regression to fit the state-price density implicitly in option pricing. Ait-Sahalia (1996) proposes a nonparametric pricing estimation procedure for interest rate derivative securities under the assumption that the unknown volatility is independent of time. Stutzer (1996) adopts the canonical valuation method, which incorporates the nonarbitrary principle embodied in the formula for calculating the expectation of the discounted value of assets under the risk-neutral probability distribution.

One of the most important nonparametric methodologies is the empirical likelihood (EL), which conducts likelihood-based statistical inference by profiling a nonparametric likelihood (see, for example, Owen 1988, 1990, 2001; DiCiccio and Romano 1989;

In this paper, we implement the EL method to price derivatives or options under risk-neutral measures. First, we construct an empirical probability constraint using historical holding period return time series observations without assuming the distribution family of the returns. Further, we view the derivative/option price directly as the parameter of interest in the EL optimization procedure. An EL-based estimate of the parameter (eg, call option price) is obtained, and the asymptotic properties of the EL ratio are studied. We further introduce a blockwise EL procedure for weakly dependent processes. Monte Carlo simulation and the empirical results for Standard & Poor’s 500 (S&P 500) index options are discussed.

The remainder of this paper is organized as follows. Section 2 provides a detailed EL procedure in option pricing. Asymptotic properties are discussed and a robust confidence interval is constructed. Section 3 demonstrates the empirical performance of EL option pricing, including both Monte Carlo simulation and S&P 500 index options. Section 4 concludes the paper with discussions.

2 EMPIRICAL LIKELIHOOD IN OPTION PRICING

Let $P(t)$ be the underlying asset price at time $t$; let $D(t)$ be the future dividend at time $t$; let $r(s,t)$ be the gross risk-free interest rate during time $s$ and $t$, with $r(t,t) = 1$; let $\mathcal{P}$ be the physical probability measure; and let $\mathcal{Q}$ be the risk-neutral probability measure (see Huang and Litzenberger 1988), under which the price process plus the accumulated dividends are martingales after normalization if no arbitrage exists in the pricing systems. To be specific, the latter leads to the following pricing formula:

$$P(t) = E^\mathcal{Q} \left[ \frac{P(T) + \sum_{s=t}^{T} D(s)r(s, T)}{r(t, T)} \right]$$

$$= E^\mathcal{P} \left[ \frac{P(T) + \sum_{s=t}^{T} D(s)r(s, T)}{r(t, T)} \frac{d\mathcal{Q}}{d\mathcal{P}} \right]. \quad (2.1)$$

Here, $d\mathcal{Q}/d\mathcal{P}$ is the Radon–Nykodym density of the marginal measure. One can price an option or a derivative security by evaluating the expected discounted value under $\mathcal{Q}$. For example, the call option price with strike price $K$ and expiry date $T$ is given by

$$C(t, T) = E^\mathcal{Q} \max[P_T - K, 0]. \quad (2.2)$$
The following subsection illustrates the idea of estimating \( C(t, T) \) through EL coupled with the change-of-measure constraint.

### 2.1 The estimating procedure

Suppose historical data is available in the format

\[
\{(P(t), D(t)), \ t = -1, -2, \ldots, -H\}.
\]

A nonparametric way of estimating the option price could be built on approximating \( Q \) by a discrete distribution supported on the observed value of the option price; namely, \( \text{HPR}(-i - T, -i)/r(-i - T, -i) \), \( 1 \leq i \leq H - T \), with the corresponding probability denoted by \( \pi_i \). Here, \( \text{HPR}(s, t) \) is the holding period return between times \( s \) and \( t \). If there is no dividend, \( \text{HPR}(-i - T, i) = P(-i)/P(-i - T) \). Then, (2.1) can be approximated by

\[
1 = \sum_{i=1}^{H-T} \frac{\text{HPR}(-i - T, -i)}{r(-i - T, -i)} \pi_i. \tag{2.3}
\]

Correspondingly, we can estimate the option price by approximating (2.2) by

\[
\hat{C}(t, T) = \sum_i \frac{\max[P_i(T) - K, 0]}{r(t, T)} \pi_i. \tag{2.4}
\]

Note that the choice of \( \pi_i \) subject to (2.3) is not unique. Stutzer (1996) uses the idea of maximum entropy, namely maximizing \( \sum_{i=1}^{H-T} \pi_i \log \pi_i \) subject to (2.3). Here, we adopt the EL method (Owen 1988) by changing the objective function from entropy to EL, namely maximizing \( \sum_{i=1}^{H-T} \log \pi_i \). This objective function can be easily interpreted as a nonparametric loglikelihood function; hence, the whole optimization procedure in our method can be interpreted as a maximum likelihood method, which is considered more efficient than a maximum entropy method. Moreover, Baggerly (1998) proposes a general class of EL-type methods, which contains both \( \sum \log \pi_i \) and \( \sum \pi_i \log \pi_i \) as special cases. In addition, Baggerly (1998) proves the EL used in this paper is the only method in the general class that has a higher-order correction of the large sample properties. We refer to Kitamura (1997) for a more detailed form of the higher-order correction. Meanwhile, noting that the sequence \( \text{HPR}(-i - T, -i)/r(-i - T, -i) \), \( 1 \leq i \leq H - T \), possesses a reasonable amount of dependence, we suggest adopting the blockwise version of the algorithm as follows.

Group the data into \( Q \) blocks, where length \( M \) is the length of the moving block. Set \( L \) to be the step size of the moving block. We obtain block weight \( \pi_i^* \) by maximizing \( \sum_{i=1}^{H-T} \log \pi_i^* \) subject to

\[
1 = \sum_{i=1}^{Q} \pi_i^* \left[ \frac{1}{M} \sum_{j=1}^{M} \frac{\text{HPR}(-i \ast L - j - T, -i \ast L - j)}{r(-i \ast L - j - T, -i \ast L - j)} \right]. \tag{2.5}
\]
Then, estimate the option price by

$$C = \sum_{i=1}^{Q} \left[ \frac{1}{M} \sum_{j=1}^{M} \max \left[ P_{i*L-j}(T) - K, 0 \right] \right] \pi_{i}^{*}. \quad (2.6)$$

This blocking idea is studied by Kitamura (1997), who argues that using blockwise methods offers a much better empirical performance for weakly dependent processes in moving-average noise terms. The estimation procedure in the spirit of Kitamura (1997) is slightly different,

$$\max_{C, \pi_{i}^{*}} \sum_{i=1}^{Q} \log \pi_{i}^{*}, \quad (2.7)$$

subject to constraints (2.5), (2.6) and

$$\sum_{i=1}^{Q} \pi_{i}^{*} = 1, \quad \pi_{i}^{*} > 0,$$

and the maximizing $C$ is our estimator. The estimated risk-neutral measure weights $p_{i}^{*}$ have the following form:

$$\pi_{i}^{*} = \left\{ Q \left( 1 + \gamma \left[ \frac{1}{M} \sum_{j=1}^{M} \frac{\text{HPR}(-i*L-j-T, -i*L-j)}{r(-i*L-j-T, -i*L-j)} - 1 \right] \right) \right\}^{-1},$$

where $\gamma$ is a Lagrange multiplier. These weights are similar to the Gibbs canonical probability in Stutzer (1996), because they put small weights when the rates of return of underlyings are far from risk-free returns. In addition, Peng (2015) shows that these two approaches yield the same asymptotic property. In our simulation below, we adopt the second method, since it is well known and there is an existing package for implementation. In particular, Qin and Lawless (1994) provide a Lagrangian with multipliers approach to solve the above-mentioned optimization problem. We can either apply the numerical optimization process or derive the solution, similar to Qin and Lawless (1994). For more details about the Lagrangian optimization or the basic properties of the EL procedure, see Owen (1990) and Qin and Lawless (1994).

### 2.2 Asymptotic properties

In this subsection, we discuss some basic asymptotic properties of the option price with respect to the EL process ((2.6) and (2.7)), which helps us to understand the asymptotic distribution of our estimate and conduct further inference.
**Theorem 2.1**  
Consider that
\[
f(HPR_t, C) = \left( \frac{\max [P_i(T) - K, 0]}{r(t, T)} - C, \frac{HPR(-t - T, t)}{r(t - T, t)} - 1 \right)^T,
\]
and further assume that

(i) the derivative price \( C \) is in a compact set \( \Theta \);

(ii) \( C_0 \) is a unique solution of \( E(f(HPR_t, C)) = 0 \);

(iii) for sufficiently small \( \delta > 0 \) and \( \eta > 0 \),
\[
E \left[ \sup_{C^* \in \Theta(C, \delta)} \| f(HPR, C^*) \| \right] < \infty
\]
for all \( C \in \Theta \);

(iv) if a sequence of \( C_j, j = 1, 2, \ldots, \) converges to some \( C \) as \( j \to \infty \),
\( f(HPR_t, C_j) \) converges to \( f(HPR_t, C) \) for all \( HPR_t \) except on a null set, which may vary with \( C \);

(v) \( C_0 \) is an interior point of \( \Theta \);

(vi) \( \text{Var}(H^{-1/2} \sum_{i=1}^H f(HPR_i, C_0)) \to S > 0 \); and

(vii) for a blockwise EL approach, we further assume the weak dependent condition
\[
\sum_{k=1}^\infty \alpha_k(k)^{1-1/d} < \infty \text{ for some constant } d > 1.
\]

We also require additional assumptions:
\[
E \| f(HPR_t, C_0) \|^{2d} < \infty, \text{ for } d > 1,
\]
\[
E \sup_{C^* \in \Theta(C_0, \delta)} \| f(HPR_t, C^*) \|^{2+\epsilon} < K, \text{ for some } \epsilon > 0.
\]

Then,
\[
LR_0 = 2 \sum_{i=1}^Q \log(1 + \gamma(\hat{C})^T f(HPR_i, \hat{C})) \rightarrow_{\text{dist}} \chi^2_1,
\]
where \( K \) is a finite number, \( \gamma(\hat{C}) \) is the Lagrange multiplier vector and \( Q \) is the total number of states. Particularly for the nonblockwise EL case (i.e., (2.6)), \( Q = H - T \).

Theorem 1 provides an asymptotic distribution of the likelihood ratio \( LR_0 \), which can be further applied to inference of the estimate. We omit the detailed proof here.\(^1\)

For independent observations of \( HPR_i \), we require only the assumptions (i)–(vi) to

\(^1\) Our proof is a direct consequence of Theorems 1 and 2 in Kitamura (1997).
have the asymptotic property of the likelihood ratio; for weak-dependent observations of HPR_i, assumption (vii) is also required. Given the simple fact that the chi-squared distribution is the square of a normal distribution, the distribution of the errors measured by the likelihood ratio will be close to the white noise when sample size goes to infinity. This means that our estimator will eventually capture almost all of the information in the data.

3 EMPIRICAL RESULTS

In this section, we first compare our method with several popular option pricing models through Monte Carlo simulation, and then conduct an empirical analysis on the option pricing for the S&P 500 index call options.

3.1 Monte Carlo simulation

3.1.1 Black–Scholes model

Following Hutchinson et al (1994), Ait-Sahalia and Lo (1995) and Stutzer (1996), we generate a geometric Brownian motion process with a 10% drift and 20% annualized volatility. First, we simulate two years of historical daily stock returns with 253 × 2 = 506 observations. We repeat this for 200 samples. For each sample, three different prices are calculated:

1. the estimated price by the EL option pricing procedure;
2. the estimated price by the Black–Scholes model with historical volatility; and
3. the actual price by the Black–Scholes model with actual volatility.

The performances of the first two prices are compared based on the mean absolute percentage error (MAPE) with respect to the third price, which is considered to be the true price. The comparison is made at different price-to-strike-price ratios (ie, \( P/X = \frac{9}{10}, 1, \frac{9}{8} \)) and different expiration dates (ie, \( T = \frac{1}{13}, \frac{1}{4}, \frac{1}{2} \)).

Table 1 provides the simulation performance: panel (a) reports the MAPE of the EL option price, and panel (b) reports the MAPE of the historical volatility-based Black–Scholes price (Hist Var). In a perfect Black–Scholes world, the Black–Scholes formula using historical volatility outperforms the EL option pricing methodology. This is because the Black–Scholes formula only requires second moment information, and 506 observations can provide a very good estimate of the second moment; the EL method, meanwhile, automatically captures the higher-order moment information, which is not beneficial to pricing options in a perfect Black–Scholes world.

We are also interested in the accuracy of the EL option pricing for different moneyness and days to maturity. The EL option pricing method gives a very good performance in pricing the in-the-money (ITM) options with small MAPE; however, the
The mean absolute percentage error (MAPE) of the EL option price to the ideal Black–Scholes price (panel (a)), and the historical volatility-based Black–Scholes price to the ideal Black–Scholes price (panel (b)) for different combinations of the relative exercise prices ($P/K$) and time to expiration date. The price dynamics follow the geometric Brownian motion, with $\mu = 0.1$ and $\sigma = 0.2$. The relative exercise prices ($P/K$) are chosen as in Rubinstein (1985) and Stutzer (1996). The time to expiration dates are 1/13, 1/4 and 1/2 years.

MAPE is very significant for out-of-the-money (OTM) options. The at-the-money (ATM) option pricing error is in between. However, the pricing errors have different patterns for ITM, ATM and OTM options. For ITM and ATM options, the fewer days to maturity, the smaller the pricing errors. For OTM options, the fewest days to maturity case has the largest pricing error, with a possible reason being that the price magnitude of the OTM options with very few days to maturity is already very small.

### 3.1.2 Stochastic volatility jump model

Bates (1996) adds a compound Poisson process to the Heston stochastic volatility model to account for the rare sudden drift of some financial assets. The stochastic processes are defined as follows:

\[
\frac{dS}{S} = \mu \, dt + \sqrt{V} \, dZ + k \, dq,
\]

\[
dV = (\alpha - \beta V) + \sigma_v \sqrt{V} \, dZ_v,
\]

\[
cov(dZ, dZ_v) = \rho \, dt,
\]

\[
P(dq = 1) = \lambda \, dt,
\]

\[
\ln(1 + k) \sim N(\log(1 + k), \delta^2).
\]
TABLE 2 MAPE of the four methods under the Bates model.

<table>
<thead>
<tr>
<th>Jump parameters</th>
<th>EL</th>
<th>Stutzer</th>
<th>Historical Black–Scholes</th>
<th>Heston model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa = -0.001, \delta = 0.019 )</td>
<td>0.3680</td>
<td>0.3804</td>
<td>0.5734</td>
<td>( 2.428 \times 10^{-7} )</td>
</tr>
<tr>
<td>( \kappa = 0.1, \delta = 0.5 )</td>
<td>0.3759</td>
<td>0.4448</td>
<td>0.6062</td>
<td>0.4765</td>
</tr>
</tbody>
</table>

Here, we use the parameter estimates in Bates (1996) to produce simulated stock prices and European option prices. We compare our nonparametric option pricing method with that of Stutzer (1996) as well as the historical Black–Scholes and Heston models. We summarize our results in Table 2. The \( \kappa \) and \( \delta \) are the mean and standard deviations of the sizes of jumps. From Table 2, we can see that when jump sizes are small, as is the case in the second row, our method beats the historical Black–Scholes model, but it loses to the Heston model. This is because when jump sizes are sufficiently small, the Bates model is extremely close to the Heston model, and, hence, calibration of the Heston model is more or less the same as using a parametric method with the true likelihood function. We know that the parametric likelihood method always achieves the lowest error bound when we use the right likelihood functions. When the jump sizes are large, however, as is the case in the third row of Table 2, our nonparametric method not only outperforms the other methods, but also performs consistently well, whether the jump sizes are large or small.

3.2 S&P 500 index options

We also implement the EL option pricing method in pricing S&P 500 index options. The daily return data is from the Center for Research in Security Prices (CRSP) and the option data is from OptionMetrics. The daily return data is from January 2011 to December 2012. We use daily return data from 2011 as our formation period and test its performance against daily index options pricing from 2012, comparing our results with the historical volatility-based Black–Scholes model and the true values. We only keep the options that have moneyness closest to 1 and days to maturity between 15 and 50.

Figure 1 shows the time series of the option prices. The red line is the true value of the market daily close price, the green line is the EL option price, the black line is the Black–Scholes option price using historical volatility, the blue line is the Heston stochastic volatility option pricing using least-squares calibration and the purple line is the method from Stutzer (1996). Due to the stock price movement, the true option prices vary from 1.5 to 3.7; however, the historical volatility-based Black–Scholes
The time series of three S&P 500 index option prices. We only keep the options that have moneyness closest to 1 and days to maturity between 15 to 50. The red line is the true value of the market daily close price, the green line is the EL option price, the black line is the Black–Scholes option price using historical volatility, the blue line is Heston stochastic volatility option pricing using least-squares calibration and the purple line is the method from Stutzer (1996).

Option prices are consistently overpriced for the ATM call options, as is documented in Hull and White (1987). In contrast, our EL option prices are closer to the true option market prices. This is because our methodology also captures the high-order moment information in the EL procedure, while the historical volatility-based Black–Scholes option model only captures the second moment information.

4 CONCLUSION

In this paper, we introduced an EL method to price derivatives under a risk-neutral measure. Based on Monte Carlo simulations and S&P 500 index option data, we showed that our method outperforms classical alternative models (Black–Scholes, Heston and Bates), thanks to our advantage in capturing higher-order moment information.
DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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Does higher-frequency data always help to predict longer-horizon volatility?

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ABSTRACT

When it comes to forecasting long-horizon volatility, multistep-ahead iterated forecasts using higher-frequency data can be more efficient than one-step-ahead direct forecasts using lower-frequency data. However, small violations of model specification in either the volatility or expected return models are compounded in the forward iteration and temporal aggregation for the higher-frequency model. In this paper, we show that realized conditional autocorrelation in return residuals is a strong predictor of the relative performance of different frequency models of volatility. When the conditional autocorrelation is high, the higher-frequency model performs markedly worse than its lower-frequency counterpart. Empirically, we show that residual autocorrelation exists in the broad cross-section of stocks at any given point in time, and that this misspecification can substantially decrease the prediction performance of higher-frequency models. Comparing the monthly volatility predictions using daily and monthly data, we show a trade-off between the gains from higher-frequency data and the susceptibility of its multistep-ahead iterated forecasts to model misspecification.

Keywords: long-horizon volatility; iterated forecasts; temporal aggregation; model selection; risk management; mixed data frequency.
1 INTRODUCTION

As trading speeds increase and high-frequency data becomes more readily available, investors seek to incorporate this higher-frequency data into their valuations and risk management. Both these procedures typically involve forecasting the volatility of various assets. Although second, millisecond and even nanosecond return data exists, financial practitioners still use longer-horizon volatility forecasts for risk control. This may be due to both laws and internal controls that require longer-horizon forecasts for some assets because of their illiquidity.¹ There are multiple ways to create longer-horizon volatility predictions. For monthly forecasts, one “iterated” approach is to estimate a daily frequency model and make a multistep-ahead forecast, while a “direct” approach is to fit a monthly model and simply make a one-step-ahead forecast. As discussed in Marcellino et al (2006), deciding which approach is better is an empirical matter: iterated forecasts are more efficient if the one-period-ahead model is correctly specified, but direct predictions are more robust to model misspecification. For a long-horizon volatility estimate, we demonstrate that the multistep-ahead forecast also suffers from the temporal aggregation.

In this paper, we show that, given a particular forecasting horizon, there is a fundamental trade-off between more data observations and higher vulnerability to mean misspecification. At first pass, it seems that having more data observations should unequivocally lead to better-performing models. However, while volatility predictions using higher-frequency data usually outperform those using lower-frequency data, higher-frequency models are more susceptible to residual mean misspecification. In fact, longer-horizon models can be more susceptible in periods with more realized misspecification in the return process. Financial practitioners should be cognizant of the trade-offs illustrated in this paper when deciding between different data and model frequencies. To assist with this decision, we show that realized sample autocorrelation provides a useful statistic with which to capture this trade-off among various models.

To isolate the effects of just the data frequency and misspecification, we keep the number of parameters in each model and the forecasting horizon constant, and then compare the performance of models that use different horizons of the same data set. In comparison with monthly models, daily models using the same data range have about twenty-two times more observations, resulting in more precise parameter estimates. However, when forming monthly forecasts, the daily prediction errors are cumulated across twenty-two periods. In this aggregation, the assumption of independent returns becomes more binding. Small temporary violations of the independence assumption get magnified. When the misspecification is sufficiently

¹ For example, in the banking sector, new rules drafted in Basel III include a clause for risk management of up to thirty days.
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large, the daily model becomes less precise than the monthly model. This is because the monthly model is better able to accommodate autocorrelations in the daily returns. At the daily level, an autocorrelation of 0.8 becomes $0.8^{22} = 0.01$ in twenty-two days, so this type of misspecification is more benign. Although the daily model outperforms the monthly model unconditionally, in the face of the trade-off between the amount of data available for the daily models and the accumulation of mean misspecification, we show that, in fact, the weekly model outperforms both models alone. Since even an autocorrelation of 0.8 becomes less than 0.1 after five trading days, the weekly model is able to get some of the benefits of more data as well as less susceptibility to residual autocorrelation.

Simulations calibrated to the data help to explicitly show the trade-off between the number of data observations and misspecification. They also provide ranges of sample autocorrelations for which monthly models outperform daily models. Because of the flexibility in the simulation approach, all our primary results also extend to other time scales. The simulation approach allows us to characterize the exact return misspecification and abstract from other potential misspecification issues that may exist in the data, such as violations of the normality assumption. For simplicity, we use the daily and monthly versions of the GARCH(1,1) model to produce monthly out-of-sample forecasts. We focus on ARCH-type models, specifically GARCH(1,1), since they have been shown by Hansen and Lunde (2005) to perform relatively well, even when their more-sophisticated counterparts are taken into consideration.

Empirically, we show that the sample autocorrelation across returns in one month is a sufficient statistic to demonstrate the difference in relative performance of daily to monthly models. We show that when daily realized sample autocorrelations are larger than about 0.5, monthly models outperform daily models; this is consistent with predicted patterns from simulations. Moreover, potential temporary daily return misspecification is a valid concern. Even though most stocks traded in the United States have unconditional daily return autocorrelations that are close to zero, Figure 1 shows that actual realized sample daily autocorrelations within each month are not all identically zero. In the cross-section of the universe of publicly traded stocks in the United States from 1990 to 2015, in any given month there are stocks whose realized sample autocorrelations are sufficiently large that a monthly horizon model outperforms a daily horizon model when generating monthly volatility forecasts.

1.1 Connection to the literature

Since the relationship between a stock’s risk and expected return is one of the fundamental questions in financial economics, it is no surprise that the topic has been widely

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2 GARCH: generalized autoregressive conditional heteroscedasticity.
At any given point in time, a substantial fraction of stocks have autocorrelations larger than 0.3 in magnitude, with most of those being negative.

studied. While we benefit greatly from the extant literature on iterated versus direct forecasting, volatility prediction, volatility measurement and temporal aggregation, our study contains an important novelty: a diagnostic tool to compare different methods that work both in theory and empirics. In addition to showing that core insights from the iterated and direct forecasting literature in the first-moment processes also apply to the second-moment models used in practice, we argue that there are thresholds based on the realized autocorrelation in the data for which one method dominates another.

The insights from our study also provide an explicit explanation as to why mixing different horizons of the same data set may be beneficial, which has been documented in the literature. Most notably, Zhang et al (2005) and Zhang (2006) propose the multiscale realized volatility (MSRV) estimator, which combines realized volatilities computed at more than two return frequencies of data. Curci and Corsi et al (2012) provide a similarly realized volatility estimator based on a combination of the multiscale regression and discrete sine transform approaches. In addition, Ghysels et al (2006) develop a method called mixed-frequency data sampling (MIDAS) to predict realized volatility from one week to four weeks; they find it outperforms the Andersen et al (2003) short-horizon realized volatility model, which uses high-frequency data. However, the reason for improvement may not have been apparent. To this end, our findings on the trade-off between misspecification and parameter estimate precision rationalize these results.
In providing an explanation for why mixing models of different frequencies may be useful, our study also fits directly into the literature of comparing “iterated” multistep-ahead time series forecasts and “direct” one-step-ahead forecasts (see Cox 1961; Weiss 1991; Tiao and Tsay 1994; Tiao and Xu 1993). Considering multiple frequencies of a single time series has been an area of active research. However, where prior research has focused primarily on the performance of models for the mean process, we study the prediction performance of models for the volatility process. Despite that difference, the fundamental trade-off between estimation precision and susceptibility to misspecification still applies. For example, Marcellino et al (2006) compare empirical iterated and direct predictions from linear time series models and find the iterated predictions have a better overall performance. McElroy (2015) further explores whence the benefits of iteration arise and finds that the method of fitting also matters.

Given our focus on volatility prediction, this study also contributes to the vast literature on implementing different volatility models. Within the more recent volatility forecasting research, Brownlees et al (2011) provide a guide to model implementation by considering estimation horizon, relevant loss functions and estimation frequency. However, they do not consider the frequency of data to use. To explore this facet, we follow the “best practice” guidelines for GARCH implementation and evaluate monthly volatility predictions from different frequency versions of the same model. For other volatility models, Poon and Granger (2003), Poon (2005) and Andersen et al (2006) provide comprehensive reviews of the volatility forecasting literature, including other forecasting methods for which the trade-off between estimation and prediction errors still applies. Our work complements the existing literature on higher-frequency volatility forecasting techniques, such as those by Aït-Sahalia and Mancini (2008), Andersen et al (2003) and Bandi and Russell (2008). Building on Anatolyev and Tarasyuk (2015), who conclude based on closed-form analytical solutions that misspecification of the mean process in an ARCH model does not distort estimates of true ARCH parameters by much, we extend the consideration of mean misspecification on the relative rankings of daily and monthly models for GARCH processes, for which there are no closed-form expressions.

We combine issues of temporal aggregation and forecasting performance using simulations to rationalize the empirical regularities found in the data. For the most part, the volatility forecasting literature for monthly horizon volatility has been less

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4 Studies focusing on the extension, estimation and implementation of these higher-frequency realized volatility models include Corsi (2009), Engle and Gallo (2006), Deo et al (2006) and Shephard and Sheppard (2010).
active than that for high-frequency volatility. In particular, the temporal aggregation of volatility literature, which includes Meddahi et al (2004) and Zaffaroni (2007), often explores how different stochastic volatility models can generate long-memory volatility processes for aggregated returns. However, this research is typically not directly focused on forecasting.

The rest of the paper is as follows. Section 3 develops the framework and shows simulation results. Section 4 shows empirical evidence of the trade-off between the number of data observations and sensitivity to misspecification. Section 5 concludes.

2 MEAN MISSPECIFICATION AND LONG-HORIZON VOLATILITY FORECASTS

To facilitate our simulation and empirical studies, we first provide a general framework for the type of misspecification we consider. On the one hand, the direct prediction of the one-step-ahead approach is straightforward to estimate a monthly volatility model. On the other hand, the multistep-ahead iterated approach requires additional assumptions to aggregate the volatilities through time. In the case of daily ARCH models, the multistep-ahead iterated prediction requires iterating upon the autoregression structure to obtain the multiperiod forecast. However, for GARCH models, we cannot iterate the process without assuming that all future GARCH terms up to the forecast are zero. This is equivalent to using an aggregating rule that all daily returns are independent in the next month. Therefore, given all future daily forecasts for all days in the month ahead, we can simply add them up to obtain a monthly aggregate forecast.

To be more precise about the misspecification admitted in our simulation experiments, we provide a general framework that illustrates how mean model misspecification of expected returns affects the relative performance of the iterated versus direct forecasting methods. In general, volatility forecasting requires a joint estimation of the conditional mean and conditional variance. Any misspecification of the conditional mean model generates misspecified realized shocks. Those realized shocks may still contain serial dependence that biases the long-horizon volatility forecasts. Since we do not take a stance on the correct mean model, the discussion in this section does not rely on specific model forms for the returns-generating process or the conditional volatility model. We want to show that there are many reasons for the misspecification of fitting volatility models.

Suppose that the continuously compounded returns are generated according to the conditional function $f_t(\cdot)$ of a set of state variables $X_t$ at time $t$:

$$r_t = f_t(X_t, X_{t-1}, \ldots) + \varepsilon_t,$$

5 Notable exceptions include Hansen et al (2011), who incorporate realized volatility measures into a GARCH framework.
where $\varepsilon_t$ are independent across time. In this setup, the conditional function $f_t(\cdot)$ captures all the time dependence in the returns process. However, a conditional model with arbitrary time-varying constraints is impossible to estimate without more assumptions. Due to statistical constraints in estimation, practitioners and researchers must instead rely on an unconditional function $\hat{f}(\cdot)$ to approximate the conditional function $f_t(\cdot)$. For example, this can be done by specifying an unconditional linear factor model, such as a capital asset pricing model (CAPM) or a Fama–French three-factor model, where risk-factor loadings are constant. Estimating $\hat{f}(\cdot)$ also produces the estimates of the realized shocks $\hat{\varepsilon}_t$ according to

$$\hat{\varepsilon}_t = r_t - \hat{f}(X_t, X_{t-1}, \ldots).$$

Because of the misspecification, the estimated residuals may be autocorrelated due to an unconditional approximation to the conditional true mean model.

To simplify the framework, we assume that the residual autocorrelation structure can be fully captured in an AR(1) form, where the autoregressive term is constant for some length of time. In the simulations for daily data, we will assume that the autoregressive term for returns across days is constant for a month. This assumption enables us to get estimates for the autoregressive term but still allows us to study the effects of time-varying mean model misspecification. Therefore, given a particular month $M$, the AR(1) setup is

$$\hat{\varepsilon}_t = \rho_M \hat{\varepsilon}_{t-1} + \eta_t$$

for the daily index $t \in M$. Here, only $\eta_t$ are independent across time, but $\hat{\varepsilon}_t$ are serially autocorrelated. In practice, the daily autoregressive term may only be constant for a much shorter period of time than one month, so empirical estimates of the autocorrelation for daily returns in one month may be subject to the actual daily autocorrelations changing through the month as well as estimation error.

The autocorrelations may arise for multiple reasons. For example, missing serially correlated macroeconomic factors or microstructure effects such as the bid–ask bounce may introduce residual autocorrelations into the estimated residuals, depending on the frequency of interest. Empirically, we show evidence of substantial residual autocorrelations in the cross-section of stocks at any given point in time. We also show that even focusing within one stock, the twenty-two-day rolling realized autocorrelation estimates vary significantly through time.

Moreover, in this setup, jointly estimating both the mean and volatility equations may not resolve the misspecified residual issue, because the unconditional mean model used may still be incorrectly specified. In practice, the misspecification cannot be entirely solved by including more terms in the estimating function $\hat{f}(\cdot)$, because we never know the true data-generating process. Therefore, this type of misspecification always remains a possibility.
We consider the consequences of this kind of misspecification on long-horizon forecasts on the iterated and direct approaches to estimating long-horizon volatility. If our target is a monthly volatility forecast that requires temporal aggregation of daily forecasts, the former method is more susceptible to the misspecification. Given the mean and volatility models structure, the cumulative $T$-step ahead continuously compounded return made under the assumption of independent residuals is

$$\hat{\varepsilon}^T = \sum_{t=1}^{T} \hat{\varepsilon}_t,$$

which requires forecasting individual $\hat{\varepsilon}_t$ for $t = 1, \ldots, T$. The time $\tau$ conditional variance of that $T$-step compounded continuous return, temporally aggregated, is

$$\text{var}_\tau(\hat{\varepsilon}^T) = \sum_{t=1}^{T} \text{var}_\tau(\hat{\varepsilon}_t) + 2 \sum_{i=1}^{T} \sum_{j=2}^{T} \text{cov}_\tau(\hat{\varepsilon}_i, \hat{\varepsilon}_j),$$

where the $g(\rho)$ function captures all the pairwise covariances from the set of $t = 1, \ldots, T$ returns and increases with larger $|\rho|$, the magnitude of autocorrelation between daily residuals. This property simply follows from the fact that the sum of the variance does not equal the variance of the sum when the autocorrelation is nonzero. The first term captures the standard temporal aggregation procedure that assumes independence in the $\hat{\varepsilon}_t$. There is also a bias term, represented by $g(\rho)$, whose size increases with the forecast horizon $T$ as well as the magnitude of $\rho$. Since we cannot statistically identify the true $\rho$, researchers will mistakenly only consider the first component on the right side of the equation through standard temporal aggregation under no serial correlation. Therefore, the serial correlation contaminates the temporal aggregation of high-frequency model predictions in forecasting long-horizon volatility.

Moreover, where Anatolyev and Tarasyuk (2015) analytically calculate the effects of mean misspecification on estimates of volatility for ARCH models, the nonlinearities of the GARCH process severely complicate any analytical attempt at temporal aggregation. Because of this, we rely on simulations to provide us with insight and the threshold at which misspecification leads one method to dominate the other.

3 MONTE CARLO SIMULATIONS

To explicitly study the impact of the mean misspecification on the iterated and direct methods of volatility forecasting, we design a simulation to compare the prediction...
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In order to see the effect of the bias term, this is defined as

$$\text{bias} = \sum_{t=1}^{T} g(\rho).$$

For higher-frequency models, we temporally aggregate daily volatilities to monthly volatilities by assuming predictions are temporally independent. In the simulations, the only differences in relative performance of the daily and monthly models are due to the conditional autocorrelation $\rho$, which is assumed to be constant across days within a month but vary across months. The simulation results show that high-frequency models do not necessarily outperform low-frequency models, especially when the conditional autocorrelation is high.

3.1 Data-generating process

To stay as close to reality as possible, we calibrate the data-generating process used in the calibration to real data. Let $t$ be the time index for a day, $t_w$ be the time index for a week and $t_m$ be the time index for a month. That is,

$$r_{t_w=1}^W = \sum_{t \in t_w=1}^5 r_t = \sum_{t=1}^5 r_t$$

and

$$r_{t_m=1}^W = \sum_{t \in t_m=1}^{20} r_t = \sum_{t=1}^{20} r_t.$$

For convenience, we use twenty days as the month length in our simulation. To get parameter values for the daily returns process $r_t$, we estimate a constant mean and GARCH(1,1) daily model using Standard and Poor’s 500 (S&P 500) daily returns from January 1, 1950 to January 31, 2016:

$$r_t = \mu + \hat{\varepsilon}_t,$$

$$\sigma_t^2 = \hat{\omega} + \hat{\alpha} \hat{\varepsilon}_{t-1}^2 + \hat{\beta} \sigma_{t-1}^2,$$

which will be used as the data-generating process. We also independently draw the conditional autocorrelation $\rho_{tm}$ for each month from the empirical distribution of daily return autocorrelations for each month, from the universe of all traded stocks in the United States. We allow for time-varying misspecification by having a process $\rho_{tm}$ with one value for each month. This allows the mean model misspecification to vary through time in a pattern similar to that in the observed data. The distribution of
estimated daily autocorrelations for each month has a mean of 4.3% and a standard deviation of 22.3%, meaning that the time-varying feature of $\rho_{tm}$ is important.

Using the framework from Section 2, we incorporate the misspecification $\rho_{tm}$ into the daily returns process using the procedure below.

1. If we are at the start of the month, draw $\rho_{tm}$ from the empirical distribution of $\{\hat{\rho}_{tm}\}$. If we are not at the start of the month, keep using the same $\rho_{tm}$ from before.

2. Draw $\eta_t$ from independent $N(0, \sigma^2_t)$ distributions, where $\sigma^2_t$ is specified according to the GARCH(1,1) model above.

3. Generate $\hat{\epsilon}_t = \rho_{tm}\hat{\epsilon}_{t-1} + \eta_t$.

4. Construct $r_t = \hat{\mu} + \hat{\epsilon}_t$.

In the procedure above, the conditional autocorrelation we set in the simulation is conditional on the month instead of the day, since the autocorrelation is constant within a month. The simulated data is in daily frequency, and we aggregate the data into weekly (five-day) and monthly (twenty-day) frequencies. The length of the simulation is thirty-seven months, with thirty-six months being used as the training period and one month being used as the testing period.

In the estimation part, we purposely ignore the nonzero autocorrelation in the residuals and compare the model performances. We estimate the constant mean and GARCH(1,1) model from daily, weekly and monthly return data. We estimate the model using the training data and make an out-of-sample prediction for the last month, using the testing period to evaluate prediction model performance.

This issue of residual serial correlation becomes more exaggerated as we move from the temporal aggregation of daily model prediction to monthly prediction. The weak autocorrelation diminishes quickly in the monthly model but affects the aggregation of daily models. The one-month-ahead prediction of the daily model shows an upward bias for negatively autocorrelated residual volatility aggregation and a downward bias for positive autocorrelated residual volatility aggregation.

### 3.2 Prediction performance comparison

In order to compare relative performance across different horizon models, we use mean absolute error and root mean square error statistics. The nature of the simulations also allows us to provide confidence intervals around these statistics. To calculate the measures, we use a simulation approach to construct the true volatility benchmark. We use the true parameter $\{\mu, \omega, \alpha, \beta, \rho_{t_{37}}\}$ and simulate the twenty-day-ahead return
1000 times. We then use the sample variance as the benchmark of “true” volatility. The measures, defined across the $K = 1000$ simulations, are

$$\text{MAE} = \frac{1}{K} \sum_{i=1}^{K} |\text{bias}_i|,$$

$$\text{RMSE} = \sqrt{\frac{1}{K} \sum_{i=1}^{K} \text{bias}_i^2},$$

where the bias is isolated by taking the difference between the actual volatility and the simple time-aggregated volatility.

### 3.3 Simulation results

When the realized autocorrelation is less than 0.3 in magnitude, the daily frequency model dominates its weekly and monthly counterparts; here, the weekly model is also a multistep iterated forecast, while the monthly model is a direct forecast. However, when the realized autocorrelation is larger than 0.3, the misspecification is large, and the longer-horizon models perform better relative to the daily model.

An additional nuance to consider is the temporal aggregation of GARCH(1,1) processes themselves; in fact, the aggregation of a GARCH(1,1) process into weekly and monthly frequencies is no longer a GARCH(1,1) process. Therefore, only the daily model is correctly specified for both mean and volatility equations. However, in the simulation results, we still find that a correctly specified daily volatility model does not necessarily outperform the misspecified weekly and monthly models when realized autocorrelation is greater than 0.3. Figure 2 shows the results of the performance metrics along with standard error bars from 100 000 simulations, where large autocorrelations correspond to those whose autocorrelation parameter is larger than 0.3 in magnitude. The higher-frequency model only outperforms the weekly and monthly models when its realized autocorrelation is small, although in absolute terms the errors are not that different. However, this suggests that correctly switching between the models will yield substantial relative improvements.

We also construct the predictive regression using realized $\rho$ to predict the performance comparison of models by allowing asymmetric parameters; this is specified as

$$\log \left( \frac{\text{bias}_{D,i}}{\text{bias}_{M,i}} \right) = a + b_1 \hat{\rho}_i + b_2 I_{\hat{\rho}_i > 0} \hat{\rho}_i + \eta_i.$$

The regression specification allows for differing effects of $\hat{\rho}_{tm}$ based on whether it is positive or negative. We perform mutual comparisons for all three models and plot the regression-implied model comparison in Figure 3, which shows the relative errors between two models depending on the conditional autocorrelation across 100 000
Higher-frequency daily models significantly outperform the other two for the small $\rho$ sample.

simulations. In the first panel of Figure 3, the monthly model outperforms the daily model when the regression-implied comparison log ratio is above 0. In this regression setup, all coefficients are highly significant at the 1% level for all three plots. Moreover, for the regressions of daily versus monthly (left panel) and daily versus weekly (right panel), the prediction power is as high as 18% $R^2$. This exercise proves that, with the framework developed above, the conditional autocorrelation of realized shocks is a strong predictor of relative performance across models of differing horizons.

4 EMPIRICAL EVIDENCE

Based on the intuition from the simulation results, we also show empirically that the realized daily autocorrelation serves as a useful statistic to compare the relative performance of daily and monthly models when making monthly out-of-sample forecasts. Our choice of comparing daily predictions with monthly ones is mainly due to data availability. The insights from these simulations apply for intraday data as well. In fact, since intraday returns exhibit even more autocorrelation than daily returns,
FIGURE 3 Regression-implied relative performance.

When we compare the log ratio of errors across models, we note that the relative errors cross 1, meaning the higher-frequency model underperforms the lower-frequency model when the conditional autocorrelations are large.

due to bid–ask bounce and other potential factors, returns misspecification is of even greater concern in this case.

We use split-adjusted log return data for all equities available through the Center for Research in Security Prices (CRSP) universe from 1992 to 2015; this comprises a total of 21,680 stocks, each surviving for various points in time. This data set consists of all publicly traded stocks on all public US exchanges. While the majority of stocks are on the New York Stock Exchange (NYSE), American Express (Amex) or Nasdaq, we also have some stocks from other exchanges as well that may be more illiquid, and may potentially admit more autocorrelation in the returns. Although the exchanges may differ slightly in their market structure, our results hold for all stocks as well as for individual stock exchanges.

To illustrate the empirical evidence in the cross-section of all stocks, we show the relative performance of daily and monthly models from snapshots of data based on disjoint subsamples; we do this to demonstrate the robustness across time, while maximizing the amount of possible data. From the simulation results, we know that the fundamental trade-off between data frequency and mean misspecification does not depend on any characteristic of the data apart from realized autocorrelation for each month. Specifically, the results should not be affected by long-term trends in volatility or what is happening in the aggregate economy. Therefore, we use repeated
nonoverlapping cross-sections, which allow different states of the economy to be represented in the analysis. This approach alleviates concerns that particular time-specific factors may drive the results. The repeated cross-sections use three years of training data and one year of testing data. Our testing periods are in the years 1995, 1999, 2003, 2007, 2011 and 2015, respectively. Table 1 shows the breakdown of the data and the number of stocks in each subsample.

To keep the amount of data used to estimate the models consistent, we use fifty-nine months to train the data, and evaluate a one-month-ahead out-of-sample volatility prediction for each training and testing data set pair. Within each training period, the GARCH models are estimated using a quasi-maximum likelihood estimator. Since the samples are nonoverlapping, we will refer to the training and testing data set pair by the testing year. The realized sample autocorrelations are calculated from daily return data for each month.

For each subsample, we estimate three mean models to highlight the fact that there is always residual realized autocorrelation in the data. Specifically, constant mean models, CAPM mean models and AR(1) mean models for returns are all unable to resolve the issue of realized residual autocorrelation (see Figure 4). Although the histogram of the residuals from the AR(1) model appears to be symmetric, all three return models still show significant dispersions in realized daily return autocorrelation.

We show in Table 2 that, apart from the AR(1) model in the 1995 testing period, about 20–30% of the data for all models and testing periods has realized daily autocorrelations greater than 0.3 in magnitude. This suggests that about 20–30% of the time the realized misspecification may make the monthly model outperform the daily model.

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**TABLE 1** Data sample.

<table>
<thead>
<tr>
<th>Training period</th>
<th>Testing period</th>
<th>Total</th>
<th>NYSE</th>
<th>Amex</th>
<th>Nasdaq</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>1996–1998</td>
<td>1999</td>
<td>5502</td>
<td>2277</td>
<td>467</td>
<td>2754</td>
<td>4</td>
</tr>
<tr>
<td>2000–2002</td>
<td>2003</td>
<td>5330</td>
<td>2189</td>
<td>385</td>
<td>2722</td>
<td>34</td>
</tr>
<tr>
<td>2004–2006</td>
<td>2007</td>
<td>4837</td>
<td>2046</td>
<td>363</td>
<td>2299</td>
<td>129</td>
</tr>
<tr>
<td>2008–2010</td>
<td>2011</td>
<td>3297</td>
<td>1179</td>
<td>168</td>
<td>1949</td>
<td>1</td>
</tr>
<tr>
<td>2012–2014</td>
<td>2015</td>
<td>5271</td>
<td>2081</td>
<td>290</td>
<td>2031</td>
<td>869</td>
</tr>
</tbody>
</table>

Most of the stocks come from the NYSE, Amex and Nasdaq exchanges.

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6 Huang et al (2008) discuss when to use the quasi-maximum likelihood as opposed to the least absolute deviance estimator, while Bollerslev (2013) surveys the different methods of estimating volatility.
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FIGURE 4  Distribution of $\hat{\rho}$ in testing sample across mean models.

Across all mean models, there is a large distribution of residual autocorrelation, which suggests that, at any point in time, some stock volatilities should be forecast using the daily model, and some should be forecast using the monthly model.
Between 14% and 37% of the testing period data has realized autocorrelations larger than 0.3, suggesting that for those 14–37% of stocks the volatility forecasts can be improved by using longer-horizon models.

### TABLE 3 Relative performance of daily and monthly models.

<table>
<thead>
<tr>
<th>Model</th>
<th>Const.</th>
<th>AR(1)</th>
<th>CAPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\rho}$</td>
<td>-0.955***</td>
<td>-1.550***</td>
<td>-0.504***</td>
</tr>
<tr>
<td></td>
<td>(0.076)</td>
<td>(0.086)</td>
<td>(0.077)</td>
</tr>
<tr>
<td>$\hat{\rho} \times I(\hat{\rho} &gt; 0)$</td>
<td>2.101***</td>
<td>2.349***</td>
<td>1.705***</td>
</tr>
<tr>
<td></td>
<td>(0.183)</td>
<td>(0.168)</td>
<td>(0.184)</td>
</tr>
<tr>
<td>Constant</td>
<td>-0.389***</td>
<td>-0.403***</td>
<td>-0.311***</td>
</tr>
<tr>
<td></td>
<td>(0.019)</td>
<td>(0.018)</td>
<td>(0.020)</td>
</tr>
<tr>
<td>Number of observations</td>
<td>29 226</td>
<td>29 229</td>
<td>29 232</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.006</td>
<td>0.011</td>
<td>0.003</td>
</tr>
</tbody>
</table>

$log(bias_{D,i} / bias_{M,i}) = a + b_1 \hat{\rho}_i + b_2 I(\hat{\rho}_i > 0) \hat{\rho}_i + \eta_i$. From a baseline of serial independence, an increase in the autocorrelation of 0.1 corresponds to an increase of 11.5% in the relative error of daily to monthly forecasts. *$p < 0.1$, **$p < 0.05$ and ***$p < 0.01$.

The main empirical finding in Table 3 shows that the ratio of the error in the daily to monthly models increases as the magnitude of monthly realized autocorrelation $\hat{\rho}_{itm}$ increases. Given the regression pooled across assets $i$ and testing periods $t_m$,

$$log(bias_{D,i} / bias_{M,i}) = a + b_1 \hat{\rho}_i + b_2 I(\hat{\rho}_i > 0) \hat{\rho}_i + \eta_i.$$  

Figure 5 exhibits a V-shaped relation. The regression results in Table 3 also show that the regression coefficients for both sides of the relationship are statistically significant at the 0.01 level for all three mean models. Despite this conditional performance based on realized sample autocorrelation, the regression results also imply that when misspecification is not present, ie, $\hat{\rho} = 0$, then the daily model performs about 31–40% better than the monthly model. This result corroborates the popular wisdom that, in
With large autocorrelation, both the linear and semi-parametric fits to the data show that the monthly model outperforms the daily model across all mean models considered.

general, daily GARCH models outperform monthly GARCH models even for monthly predictions.

However, the monthly model outperforms the daily model more than 10% of the time, when the realized sample autocorrelation is substantial. The log ratio of daily
to monthly performance not only increases as the sample autocorrelation increases, but also crosses zero. This means that for large values of realized autocorrelation, the monthly model outperforms the daily model. From the regression results in Table 3, a realized daily sample autocorrelation of 0.6 makes the monthly model beat the daily model by between 6% and 40%, depending on the mean model considered. Meanwhile, when the realized autocorrelation is low, the daily model outperforms the monthly model.

Figure 1 shows the evidence in the stock data; sorting the data based on realized autocorrelation affects the relative performance of the daily and monthly models. Across all stocks, the daily model outperforms the monthly model when the realized monthly autocorrelation is between $-0.6$ and $-0.25$ at the negative end, and between $0.25$ and $0.5$ at the positive end, depending on the mean model. This range covers about 90% of the data at any given time, based on the distribution of realized autocorrelations shown in Figure 1. The range estimated in the data depends on the mean model assumed, and it may be larger than the $-0.3$ to $0.3$ from simulations due to other sources of model misspecification that we have not considered in this paper. Although the coefficient for realized autocorrelation is highly statistically significant in explaining the relative errors between daily and monthly models, the $R^2$ on all the regressions in Table 3 are small.

Moreover, this empirical finding is robust to the regression specification. The results remain even when using a semi-parametric method that fits a spline to the relationship of the log performance ratio and sample autocorrelation. This is also true when individually dropping each sample period as well as sliding the definition of all subsamples back in time. The relationship between sample autocorrelation and relative performance shown here is informative when considering the trade-off we face when picking different models to forecast volatility.

5 CONCLUSION

We show that higher-frequency models are more sensitive to small violations of the independence assumption by developing intuition through simulations and showing it in the data. Since longer-horizon forecasts compound mean misspecification over more periods, they can underperform longer-horizon models, which have fewer data observations. Our simulation framework extends to various model horizons. The empirical evidence and theoretical foundation also offer an explanation for why models that use multiple-frequency data such as that shown in Ghysels et al (2006) perform well, since those models can average out the misspecification across horizons while maintaining precise model estimates from higher-frequency data. As the use of higher-frequency data becomes more widespread, this trade-off will become...
Does higher-frequency data always help to predict longer-horizon volatility?

increasingly important for practitioners to bear in mind when choosing models to forecast volatility.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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Research Paper

Optimal execution of accelerated share repurchase contracts with fixed notional

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ABSTRACT

Whether it be to take advantage of stock undervaluation or in order to distribute part of their profits to shareholders, firms may buy back their own shares. One of the ways they do this is by including accelerated share repurchases as part of their repurchase programs. We study the pricing and optimal execution strategy of an accelerated share repurchase contract with a fixed notional. In such a contract the firm pays a fixed notional $F$ to the bank and receives in exchange a number of shares corresponding to the ratio of $F$ to the average stock price over the purchase period (the duration of this period being decided upon by the bank). From a mathematical point of view, the problem is related to both optimal execution and exotic option pricing.

Keywords: optimal execution; ASR contracts; optimal stopping; stochastic optimal control; utility indifference pricing.
1 INTRODUCTION

The traditional way for a firm to repurchase its own shares is through open-market repurchase programs (OMRs). However, as reported in Bargeron et al (2011), after they have announced that they intend to buy back shares, a substantial number of firms do not commit to their initial plan. Unexpected shocks on prices or on the liquidity of the stock may indeed provide incentives to slow down, postpone or even cancel repurchase programs.

In order to make a credible commitment to buying back shares, an increasing number of firms enter accelerated share repurchase (ASR) contracts with investment banks. There are several kinds of ASR contracts, and a distinction is traditionally made between ASRs with a fixed number of shares and ASRs with a fixed notional.

In the case of an ASR with a fixed number of shares, $Q$, the contract is as follows.2

1. At time $t = 0$, the bank borrows $Q$ shares from shareholders (usually institutional investors) and gives the shares to the firm in exchange for a fixed amount, $QS_0$, where $S_0$ is the mark-to-market price of the stock at time $t = 0$. The bank then has to progressively buy back $Q$ shares on the market to return $Q$ shares to the initial shareholders, and go from a short position to a flat position in the stock.

2. The bank is long an option with payoff $Q(A_t - S_0)$ (the firm being short of this option), where $A_t$ is the average price between 0 and $t$ (in practice, the average of closing prices or the average of daily volume-weighted average prices (VWAPs)) and where $\tau$ is chosen by the bank from a set of specified dates $\tau_1, \ldots, \tau_k$.3

The firm eventually pays, on average, a price $A_t$ for each of the $Q$ repurchased shares.

In the case of a fixed-notional ASR with notional $F$, the contract is as follows.4

1. At time $t = 0$, the bank borrows $Q$ shares from shareholders (usually $Q = \alpha F/S_0$, where $\alpha \in [0, 1]$ is usually around 80%) and gives the shares to the firm in exchange for the fixed amount, $F$.

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1 Privately negotiated repurchases and self-tenders represent a small percentage of the total amount of repurchases (see Chemmanur et al 2010).

2 We present here the case of a prepaid ASR with a fixed number of shares. In many cases, especially in Europe, the shares are delivered not at inception but at the end of the contract. Although share repurchase is not accelerated in this case, the pricing and hedging of the contract are the same if we ignore funding and interest rate issues.

3 There is usually a latency period of a few weeks, after $t = 0$, during which the option cannot be exercised.

4 As above, we only discuss prepaid contracts. Post-paid contracts also exist.
(2) An option is embedded in the contract: when the bank exercises this option at date $\tau$ (the date is chosen by the bank from a set of specified dates $\tau_1, \ldots, \tau_k$), there is a transfer of $(F/A_\tau) - Q$ shares between the bank and the firm, so that the actual number of shares obtained by the firm is $F/A_\tau$.

Hence, the firm pays $F$ and eventually gets $F/A_\tau$ shares.

As above, the bank will progressively buy shares on the market in order to return $Q$ shares to the initial shareholders.

Pricing and hedging these contracts are mathematical problems that cannot be solved satisfactorily using the classical theory of derivatives pricing. At time $t = D_0$, the bank has a short position on the stock and it will buy shares on the market over a period of a few weeks to a few months. Consequently, the problem is both one of optimal execution and one of option pricing: a cash-settled exotic option with Bermudan exercise dates in the case of an ASR with a fixed number of shares, and a physically settled exotic option with Bermudan exercise dates in the case of a fixed-notional ASR. Ignoring the interactions between the two problems would lead to a suboptimal strategy and mispricing. Moreover, since the nominal/notional of these options is large, execution costs must be taken into account, and we propose a framework from the literature on optimal execution to price and optimally execute these contracts.

The problem of pricing derivative products with a large nominal has been tackled in the literature using the tools of optimal execution. Rogers and Singh (2010) considered execution costs that are not linear in (proportional to) the volume executed, but rather strictly convex, to account for liquidity effects (as opposed to the literature on pricing with transaction costs; see, for example, Leland 1985). They considered an objective function that penalizes both execution costs and mean-squared hedging error at maturity. They obtained, in this close-to-mean-variance framework, a closed-form approximation for the optimal hedging strategy of a vanilla option when illiquidity costs are small. Li and Almgren (2016), motivated by saw-tooth patterns observed on several US stocks (see Lehalle et al 2012, 2013), considered a model with both permanent and temporary effects. In their model, they assumed execution costs were quadratic and considered a constant $\Gamma$ approximation. Using another objective function, they obtained a closed-form expression for the hedging strategy of a call option. Finally, Guéant and Pu (2015) proposed a method to price and hedge a vanilla option in a utility-based framework, under general assumptions on market impact. In particular, they considered both physical settlement and cash settlement.

The ASR contract case is, however, more complex than that of vanilla options, because the payoff is not a classical European payoff, but rather an exotic one with

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5 $\alpha$ is chosen small enough that the transfer is almost always from the bank to the firm, i.e., $F/A_\tau \geq Q$. In what follows, we implicitly assume that the settlement is from the bank to the firm.
Asian and Bermudan features, and because the problem to be solved is one of both option pricing/hedging and optimal execution.

Among the different ways to buy back shares, economists have discussed the role of ASR contracts (see, for example, Bargeron et al 2011; Chemmanur et al 2010). The mathematical literature on ASR contracts is, however, rather limited and very recent. The first paper on the topic was an interesting one by Jaimungal et al (2013), who focus on ASR contracts with a fixed number of shares. For finding the optimal buying strategy, they propose a model in continuous time with the following characteristics: stock prices are modeled with a perturbed geometric Brownian motion (the perturbation accounts for permanent market impact); execution costs are quadratic as in the original Almgren–Chriss models (Almgren and Chriss 1999, 2001); the range of possible exercise dates is $[0, T]$ (the product is American rather than Bermudan). The optimal strategy is found in Jaimungal et al (2013) for an agent who is risk neutral, although Jaimungal et al add inventory penalties, as do Cartea et al (2014) and Cartea and Jaimungal (2015). The main interest of Jaimungal et al (2013) is that they manage to reduce the problem to a three-variable partial differential equation, whereas the initial problem is in five dimensions. In particular, they show that the exercise boundary depends only on the time to maturity and the ratio of the stock price to its average value since inception. The case of ASR contracts with a fixed number of shares was also considered in a different model by Guéant et al (2015). In this discrete-time model, a risk-averse agent is considered in an expected utility framework. A permanent market impact is considered as in Jaimungal et al (2013), but stock prices are assumed to be normal rather than lognormal. Also, a general form of execution cost is allowed as in Guéant (2015), and there is a finite set of exercise dates. As in Jaimungal et al (2013), the model in Guéant et al (2015) boils down to a set of equations with three variables. However, in Guéant et al (2015), the three variables are time, the number of shares to be bought and the difference between the current stock price and the average price since time $t = 0$. A novel fast tree-based numerical method is proposed in Guéant et al (2015).6

The model we propose in this paper is inspired by Guéant et al (2015). In particular, we use the same expected utility framework. However, since we focus on the specific case of fixed-notional ASR contracts, the reduction of the problem to a three-dimensional one is not possible anymore. In particular, the original tree-based method proposed in Guéant et al (2015) to numerically approximate the optimal strategy cannot be used. In Section 2, we present the framework of the model without permanent market impact. As in Guéant et al (2015), the model is in discrete time and we end up with a characterization of the optimal buying/hedging strategy with

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6 Trees are not recombinant but the number of nodes is a cubic function of the number of time periods.
Optimal execution of ASR contracts with fixed notional

recursive Bellman equations. In Section 3, we introduce these Bellman equations, along with a change of variables to reduce the dimensionality of the problem (from five to four). We also introduce the indifference price of the ASR contract. In Section 4, we present a numerical method involving trees and splines to approximate the solution of the problem, along with examples. The introduction of permanent market impact is presented in Appendix A online. Proofs are given in Appendix B online.

2 SETUP OF THE MODEL AND BELLMAN EQUATIONS

We consider the pricing and hedging of a fixed-notional ASR from the point of view of an investment bank buying back shares for a client (a firm).\(^7\) Throughout the paper, the notional of the ASR contract will be denoted by \(F\), and the maturity of the contract will be denoted by \(T\).

The model we consider is a discrete-time one where each period of time (of length \(\delta t\)) corresponds to one day. In other words, if the interval \([0, T]\) corresponds to \(N\) days, we assume that decisions are made at times \(t_0 < \cdots < t_n < \cdots < t_N\), where \(t_0 = 0\), \(t_n = n\delta t\) and \(t_N = N\delta t = T\).

We introduce a probability space \((\Omega, (\mathcal{F}_n)_{0 \leq n \leq N}, \mathbb{P})\), where the filtration satisfies the usual assumptions. All random variables will be defined on this probability space.

As far as prices are concerned, we start with an initial price \(S_0\), and we consider that the dynamics of the price is given by\(^8\)

\[
S_{n+1} = S_n + \sigma \sqrt{\delta t} \varepsilon_{n+1} \quad \text{for all } n \in \{0, \ldots, N - 1\},
\]

where \(\varepsilon_{n+1}\) is \(\mathcal{F}_{n+1}\)-measurable and where the random variables \((\varepsilon_n)_n\) are assumed to be independent and identically distributed with mean 0 and variance 1. We also assume that these variables have a moment-generating function defined on \(\mathbb{R}_+\).

Depending on the exact payoff of the considered ASR contract, \(S_n\) can be regarded as the daily VWAP over the period \([t_{n-1}, t_n], n \geq 1\), or the closing price on the \(n\)th day.\(^9\)

The average price process entering the definition of the payoff in the ASR contract is denoted by \((A_n)_{n \geq 1}\):

\[
A_n = \frac{1}{n} \sum_{k=1}^{n} S_k.
\]

\(^7\) The mechanism of the contract is explained in the previous section.

\(^8\) A linear permanent market impact as in Gatheral (2010) can be introduced in the model (see Appendix A online). However, we believe that, in the case of ASR contracts, there is little information in trades, and subsequently almost no permanent market impact, since buyback programs are usually announced before they actually occur.

\(^9\) The value of \(\sigma\) depends on what \(S\) stands for.
In order to buy shares, we assume that, every day, the bank sends an order to be executed over the day. At time \( t_n \), the size of the order sent by the bank is denoted by \( v_n \delta t \), where the process \((v_n)_n\) is assumed to be adapted. Therefore, the number of shares that are bought on the market is given by

\[
q_0 = 0, \quad q_{n+1} = q_n + v_n \delta t.
\]

The price paid by the bank for the shares bought over \([t_n, t_{n+1}]\) is assumed to be \( S_{n+1} \) plus execution costs,\(^{10}\) the latter being modeled by a function \( L \in C(\mathbb{R}, \mathbb{R}+) \) satisfying the following assumptions:

- \( L(0) = 0; \)
- \( L \) is strictly convex, even and increasing on \( \mathbb{R}+; \) and
- \( L \) is asymptotically superlinear, ie,

\[
\lim_{\rho \to +\infty} \frac{L(\rho)}{\rho} = +\infty.
\]

The cash spent by the bank is modeled by the process \((X_n)_{0 \leq n \leq N}\) defined by

\[
X_0 = 0, \quad X_{n+1} = X_n + v_n S_{n+1} \delta t + L \left( \frac{v_n}{V_{n+1}} \right) V_{n+1} \delta t,
\]

where \( V_{n+1} \delta t \) is the market volume over the period \([t_n, t_{n+1}]\). We assume that the process \((V_n)_n\) is \( \mathcal{F}_0 \)-measurable.

**Remark 2.1** In applications, \( L \) is often a power function, ie, \( L(\rho) = \eta |\rho|^{1+\phi} \) with \( \phi > 0 \), or a function of the form \( L(\rho) = \eta |\rho|^{1+\phi} + \psi |\rho| \) with \( \phi, \psi > 0 \). In other words, the execution costs per share are of the form \( \eta |\rho|^\phi + \psi |\rho| \), where \( \rho \) is the participation rate. The coefficients for this form of temporary market impact function have been estimated in Almgren *et al* (2005).

For the model to be realistic, we also assume that the bank cannot buy/sell too many shares compared with the available liquidity. In other words, we impose that \( \rho V_{n+1} \leq v_n \leq \bar{\rho} V_{n+1} \), where \( \rho \) can be of either sign.\(^{11}\)

We then introduce a nonempty set of indexes \( \mathcal{N} \subset \{1, \ldots, N - 1\} \) corresponding to the times at which the bank can choose to exercise the option (note that \( N \notin \mathcal{N} \) because the bank has no choice at time \( T \): the settlement is made at time \( T \) if it has not been made beforehand). In practice, this set is usually of the form

\[
\{n_0, n_0 + 1, \ldots, N - 1\}, \quad \text{where} \ n_0 > 1.
\]

\(^{10}\) If \( S \) models VWAP, we assume implicitly that the bank has sent a VWAP order. If \( S \) models closing prices, we implicitly assume that a target close order has been sent by the bank.

\(^{11}\) Constraints of this type are sometimes imposed in the contract.
When the bank decides to exercise the option (or at maturity if the option has not been exercised), the contract is settled. We denote by \( n^* \in \mathcal{N} \cup \{N\} \) the time index corresponding to the settlement. At time \( t_{n^*} \), the bank delivers \( (F/A_{n^*}) - Q \) shares to the firm and then the bank has to return to a flat position. Overall, the bank must buy \( (F/A_{n^*}) - q_{n^*} \) shares after time \( t_{n^*} \). As this problem is purely an optimal execution problem that depends only on the past through \( (q_{n^*}, A_{n^*}) \), we can assume that the cost of buying these shares is of the form

\[
\left( \frac{F}{A_{n^*}} - q_{n^*} \right) S_{n^*} + \ell \left( \frac{F}{A_{n^*}} - q_{n^*} \right),
\]

where the function \( \ell \) represents a risk-liquidity premium to account for the execution costs and the risk in the execution process after time \( t_{n^*} \) (see Remark 2.2 and Guéant (2014, 2015, 2016)).

The optimization problem the bank faces is given by

\[
\sup_{(v, n^*) \in \mathcal{A}} \mathbb{E} \left[ -\gamma \left( F - X_{n^*} - \left( \frac{F}{A_{n^*}} - q_{n^*} \right) S_{n^*} - \ell \left( \frac{F}{A_{n^*}} - q_{n^*} \right) \right) \right],
\]

where \( \mathcal{A} \) is the set of admissible strategies, defined as

\[
\mathcal{A} = \{ (v, n^*) \mid v = (v_n)_{0 \leq n \leq n^* - 1} \text{ is } (\mathcal{F})\text{-adapted}, \ \rho V_{n+1} \leq v_n \leq \bar{\rho} V_{n+1}, \ \text{and where } n^* \text{ is an } (\mathcal{F})\text{-stopping time taking values in } \mathcal{N} \cup \{N\} \},
\]

and where \( \gamma \) is the absolute risk-aversion parameter of the bank.

**Remark 2.2** To be consistent with the expected utility framework, the penalty function \( \ell \) should be linked to the indifference price of a block of \( q \) shares (which is indeed of the form \( q S + \ell(q) \)). Expressions for \( \ell \) have been given in a continuous-time model in Guéant (2015). If we assume that execution after \( t_{n^*} \) is at a constant participation rate, we can also consider for \( \ell \) the risk-liquidity premium associated with a liquidation at participation rate \( \rho \) (see Guéant 2014):

\[
\ell(q) = \frac{L(\rho)}{\rho} |q| + \frac{\gamma \sigma^2}{6 \rho V} |q|^3,
\]

where \( V \) is the average value of the market volume process. This is the function we consider throughout this paper (see also Guéant 2016).

### 3 PRICING AND OPTIMAL STRATEGY

#### 3.1 Bellman equations

Solving this problem requires us to determine both the optimal execution strategy of the bank and the optimal stopping time. To do this, we use dynamic programming.
We introduce the value functions \( (u_n)_{0 \leq n \leq N} \) defined by

\[
u_n(x, q, S, A) = \sup_{(v, n^*) \in \mathcal{A}_n} \mathbb{E} \left[ -\exp \left( -\gamma \left( F - X_{n^*}^{n, x, v} - \left( \frac{F}{A_{n^*}^{n, A, S}} - q_{n^*}^{n, q, v} \right) S_{n^*}^{n, S} \right) \right) \right],
\]

where \( \mathcal{A}_n \) is the set of admissible strategies at time \( t_n \), defined as

\[
\mathcal{A}_n = \{(v, n^*) \mid v = (v_k)_{n \leq k \leq n^* - 1} \text{ is } (\mathcal{F})\text{-adapted}, \rho V_{k+1} \leq v_k \leq \tilde{\rho} V_{k+1},
\]

\( n^* \) is an \( (\mathcal{F})\)-stopping time taking values in

\[
(\mathcal{N} \cup \{N\}) \cap \{n, \ldots, N\},
\]

and where the state variables are defined for \( 0 \leq n \leq k \leq N \) and \( k > 0 \) by

\[
X_{k}^{n, x, v} = x + \sum_{j=n}^{k-1} v_j S_{j+1}^{n, S} \delta t + L \left( \frac{v_j}{V_{j+1}} \right) V_{j+1} \delta t,
\]

\[
q_{k}^{n, q, v} = q + \sum_{j=n}^{k-1} v_j \delta t,
\]

\[
S_{k}^{n, S} = S + \sigma \sqrt{\delta t} \sum_{j=n}^{k-1} \varepsilon_{j+1},
\]

\[
A_{k}^{n, A, S} = \frac{n}{k} A + \frac{1}{k} \sum_{j=n}^{k-1} S_{j+1}^{n, S}.
\]

**Remark 3.1** In fact, when \( n = 0 \), the function \( u_n \) is independent of \( A \).

It is then straightforward to prove that the value functions are characterized by the recursive equations of Proposition 3.2.

**Proposition 3.2** The family of functions \( (u_n)_{0 \leq n \leq N} \) defined above is the unique solution of the Bellman equation:

\[
u_n(x, q, S, A)
\]

\[
\begin{cases}
-\exp \left( -\gamma \left( F - X_{n^*}^{n, x, v} - \left( \frac{F}{A_{n^*}^{n, A, S}} - q_{n^*}^{n, q, v} \right) S_{n^*}^{n, S} \right) \right) & \text{if } n = N, \\
\max \left\{ \bar{u}_{n, n+1}(x, q, S, A), \right. \\
\left. -\exp \left( -\gamma \left( F - X_{n^*}^{n, x, v} - \left( \frac{F}{A_{n^*}^{n, A, S}} - q_{n^*}^{n, q, v} \right) S_{n^*}^{n, S} \right) \right) \right\} & \text{if } n \in \mathcal{N}, \\
\bar{u}_{n, n+1}(x, q, S, A) & \text{otherwise},
\end{cases}
\]
where
\[
\tilde{u}_{n,n+1}(x, q, S, A) = \sup_{\rho V_{n+1} \leq \tilde{\rho} V_{n+1}} \mathbb{E}\left[u_{n+1}\left(X_{n+1}^n, q_{n+1}^n, S_{n+1}^n, A_{n+1}^n, S\right)\right].
\]

Our goal is now to simplify the problem so that a solution can be computed numerically. By “solution”, we mean

(i) an optimal strategy for both optimal execution and optimal stopping, and

(ii) a price for the ASR contract.

For the second problem, we use the concept of indifference price (see below).

3.2 Toward a simpler system of equations

Following the classical economic fact that the current wealth of an agent has no impact on their decision-making process when they have a constant absolute risk aversion, we introduce the change of variables

\[u_n(x, q, S, A) = -\exp(-\gamma(-x + qS - \theta_n(q, S, A))).\]

The functions \((\theta_n)_n\) are then characterized by the recursive equations of Proposition 3.3.

**Proposition 3.3** \((\theta_n)_n\) satisfies

\[
\theta_n(q, S, A) = \begin{cases} 
F\left(\frac{S}{A} - 1\right) + \ell\left(\frac{F}{A} - q\right) & \text{if } n = N, \\
\min \left\{ \tilde{\theta}_{n,n+1}(q, S, A), F\left(\frac{S}{A} - 1\right) + \ell\left(\frac{F}{A} - q\right) \right\} & \text{if } n \in \mathcal{N}, \\
\tilde{\theta}_{n,n+1}(q, S, A) & \text{otherwise,}
\end{cases}
\]

where

\[
\tilde{\theta}_{n,n+1}(q, S, A) = \inf_{\rho V_{n+1} \leq \tilde{\rho} V_{n+1}} \frac{1}{\gamma} \log \left(\mathbb{E}\left[\exp\left(-\gamma\left(q\sigma\sqrt{\delta t} \epsilon_{n+1} - L\left(\frac{v}{V_{n+1}}\right) V_{n+1} \delta t - \theta_{n+1}\left(q + v \delta t, S + \sigma \sqrt{\delta t} \epsilon_{n+1}, \frac{nA}{n+1} + \frac{S}{n+1} + \frac{\sigma \sqrt{\delta t} \epsilon_{n+1}}{n+1}\right)\right)\right]\right].
\]
Remark 3.4 As above for \( u_0, \theta_0 \) is independent of \( A \), and we write \( \theta_0(q, S) \) instead of \( \theta_0(q, S, A) \).

This change of variables, along with the associated recursive equations, shows that the dimension of the problem can be reduced from five to four. Indeed, the variable \( x \) does not appear in the functions \( (\theta_n)_n \) anymore. Jaimungal et al (2013), and similarly Guéant et al (2015), manage, in the case of an ASR contract with a fixed number of shares, to reduce the dimension of the problem from five to three. Indeed, they respectively consider the ratio \( A/S \) and the spread \( A - S \) instead of the couple \( (A, S) \). In the case of a fixed-notional ASR, this reduction in dimension is not possible anymore because of the additional convexity in the payoff (the term \( F/A \)), and the problem to be solved is therefore in four dimensions.

To find the optimal strategy, the method is first to solve the equations for \( (\theta_n)_n \) recursively. When \( n \in \mathcal{N} \), the option is then exercised if

\[
\tilde{\theta}_{n,n+1}(q, S, A) > F\left(\frac{S}{A} - 1\right) + \ell\left(\frac{F}{A} - q\right).
\]

When the option is not exercised, the optimal strategy consists in sending an order of size \( v^* \delta t \), where \( v^* \) is a minimizer in the definition of \( \tilde{\theta}_{n,n+1} \).

### 3.3 Price and strategy: the effects at stake

In addition to a reduction in the dimension of the problem, the main interest of the change of variables introduced earlier lies in the link between the functions \( (\theta_n)_n \) and the indifference price of the ASR contract.

The indifference price of the ASR contract is, by definition, the amount of cash (positive or negative) to give to the bank so that it is indifferent between accepting and not accepting the ASR contract. Mathematically, it is defined as

\[
\Pi := \inf \left\{ p \left| \sup_{(q,v^*) \in \mathcal{A}} \mathbb{E} \left[ -\exp\left( -\gamma \left( p + F - X_{n*}^{0,0,v} \right) \right) \right] \right\} \geq -1 \right\}.
\]
The link between the price and the functions \((\theta_n)_n\) is given by the following proposition.

**Proposition 3.5**

\[ \Pi = \theta_0(0, S_0). \]

When \(\Pi\) is positive, it means that the execution costs and the cost of the risk borne by the bank are larger than the potential gain for the bank associated with the option embedded in the contract. On the other hand, if \(\Pi\) is negative, the bank values the option sufficiently highly to compensate the execution costs and the risk associated with the execution strategy.

**Remark 3.6** In practice, when a firm wants to buy back shares, competition occurs between banks willing to enter the deal (banks for which \(\Pi \leq 0\)). However, instead of proposing rebates in cash that would correspond to at most \(-\Pi\), banks propose a discount \(\beta\) on the average price. The optimization problem is then

\[
\sup_{(v^*, n^*) \in A} \mathbb{E} \left[ -\exp \left( -\gamma \left( F - X_{n^*}^{0,0,v} - \left( \frac{F}{(1 - \beta) A_{n^*}^{0,S_0,S_0} - q_{n^*}^{0,0,v}} \right) S_{n^*}^{0,S_0} \right) \right. \\
\left. - \ell \left( \frac{F}{(1 - \beta) A_{n^*}^{0,S_0,S_0} - q_{n^*}^{0,0,v}} \right) \right) \right].
\]

From a mathematical point of view, this problem is very similar to the original one. However, the maximum rebate \(\beta^*\), defined as

\[
\beta^* := \sup \left\{ \beta \leq 1 \left| \sup_{(v^*, n^*) \in A} \mathbb{E} \left[ -\exp \left( -\gamma \left( F - X_{n^*}^{0,0,v} \right) \right. \\
\left. - \left( \frac{F}{(1 - \beta) A_{n^*}^{0,S_0,S_0} - q_{n^*}^{0,0,v}} \right) S_{n^*}^{0,S_0} \right) \right. \\
\left. - \ell \left( \frac{F}{(1 - \beta) A_{n^*}^{0,S_0,S_0} - q_{n^*}^{0,0,v}} \right) \right) \right) \supseteq -1 \right\},
\]

cannot be computed directly.

The functions \((\theta_n)_n\) also enable us to understand the different effects at stake when it comes to execution strategy and choice of stopping time. Let us thus expand the expression for \(\theta_n\) (we omit superscripts for simplicity). As proved in Appendix B
online, we have

$$\theta_n(q, S, A) = \inf_{(v,n^*) \in A_n} \frac{1}{\gamma} \log \left( \mathbb{E} \left[ \exp \left( -\gamma \left( \sum_{j=n}^{n^*-1} \left( q_j - \frac{j}{n^*} \frac{F}{A_n^*} \right) \epsilon_{j+1} + \frac{n^*}{n^*} (A - S) \frac{F}{A_n^*} \right) \right. \right. \right.$$

$$\left. \left. \left. \sum_{j=n}^{n^*-1} \left( \frac{V_j}{V_{j+1}} \right) V_{j+1} \delta t - \ell \left( \frac{F}{A_n^*} - q_n^* \right) \right) \right] \right).$$

To understand the different effects, we have to distinguish between what happens before and after the option is exercised.

Before time $t_{n^*}$, the bank buys back shares on the market. As for all execution problems, we have an execution cost component and a risk component. The bank pays execution costs depending on its participation rate to the market (this is “liquidity term I”), and it is also exposed to price moves. However, owing to the form of the payoff, the risk is partially hedged. When prices go up, the bank buys shares at higher prices, but the number of shares to buy ($F = A_n^*$) decreases. Alternatively, when prices go down, the bank buys shares at lower prices, but the number of shares to buy increases. Unlike what happens with ASR contracts with a fixed number of shares, the hedge is only partial in the case of a fixed-notional ASR: “risk term I” cannot be set to zero using an adapted strategy.

After time $t_{n^*}$, the execution costs are similar, but the risk is not partially hedged anymore: this is the “post-exercise risk-liquidity term” (it requires us to choose $\ell$ properly; see Remark 2.2).

The above analysis is linked to the execution process used for buying back shares. As far as the option is concerned, the bank has an incentive to exercise it when $S$ is below $A$, since $F/A$ will automatically increase as $A$ tends toward $S$. The cost of not exercising now is given by the “spread term”.

We therefore see that there is a trade-off between exercising when $S$ goes below $A$ in order to eventually deliver fewer shares, and not executing too soon in order to (partially) hedge the risk associated with the execution process. This implies accelerating the buying process when $S$ decreases, and decelerating the buying process (or even selling shares) when $S$ increases.
4 NUMERICAL METHODS AND EXAMPLES

4.1 A tree-based method

To solve the problem numerically, we introduce a tree-based method. This method uses a tree to model the diffusion of prices. For each node in the tree, indexed by a time index \( n \) and a price \( S \), we shall compute the values of the function \( \theta_n(\cdot, S, \cdot) \) on a grid.

4.1.1 Structure of the tree

We consider that innovations \( (\varepsilon_n)_{n \geq 1} \) have the following distribution:

\[
\varepsilon_n = \begin{cases} 
+2 & \text{with probability } \frac{1}{12}, \\
+1 & \text{with probability } \frac{1}{6}, \\
0 & \text{with probability } \frac{1}{3}, \\
-1 & \text{with probability } \frac{1}{6}, \\
-2 & \text{with probability } \frac{1}{12}.
\end{cases}
\]

We introduce, for \( n \in \{0, \ldots, N\} \) and \( \zeta \in \{0, \ldots, 4n\} \), the function \( \Theta_n^\zeta \) defined by

\[
\Theta_n^\zeta(q, A) = \theta_n(q, S_0 + \sigma \sqrt{\delta t} (\zeta - 2n), A).
\]

Following Proposition 3.3, and setting

\[
A'_{n,A} = \frac{n}{n+1}A + \frac{1}{n+1}(S_0 + \sigma \sqrt{\delta t} (\zeta - 2n)) + \frac{1}{n+1} \sigma \sqrt{\delta t} \varepsilon_{n+1}, \quad (4.1)
\]

the family of functions \( (\Theta_n^\zeta) \) satisfies

\[
\Theta_n^\zeta(q, A) = \begin{cases} 
F\left( \frac{S_0 + \sigma \sqrt{\delta t} (\zeta - 2n)}{A} - 1 \right) + \ell \left( \frac{F}{A} - q \right) & \text{if } n = N, \\
\min \left\{ \Theta_n^{\zeta,n+1}(q, A), \right. \\
F\left( \frac{S_0 + \sigma \sqrt{\delta t} (\zeta - 2n)}{A} - 1 \right) + \ell \left( \frac{F}{A} - q \right) \left\} & \text{if } n \in \mathcal{N}, \\
\Theta_n^{\zeta,n+1}(q, A) & \text{otherwise},
\end{cases}
\]

12 This method is different from that used in Guéant et al (2015), where nodes were indexed by the spread \( S - A \).

13 The distribution is chosen to match the first four moments of a standard normal: \( \mathbb{E}[\varepsilon_n] = 0 \), \( \mathbb{E}[\varepsilon_n^2] = 1 \), \( \mathbb{E}[\varepsilon_n^3] = 0 \) and \( \mathbb{E}[\varepsilon_n^4] = 3 \).
where
\[
\tilde{\Theta}_{n,n+1}^{\xi}(q, A) = \inf_{v \in [\rho V_{n+1}, \bar{\rho} V_{n+1}]} \frac{1}{\gamma} \log \left( \mathbb{E} \left[ \exp \left( -\gamma \left( q \sigma \sqrt{\delta t} \varepsilon_{n+1} - L \left( \frac{v}{V_{n+1}} \right) V_{n+1} \delta t \right. \right. \right. \\
+ \left. \left. \left. \Theta_n^{\xi+(\varepsilon_{n+1}+2)}(q + v \delta t, A_{n,A}^{'(\cdot)}) \right) \right) \right] \right).
\]

The functions \((\Theta_n^{\xi})_{n,\xi}\) are computed recursively on a grid. Each index \((n, \xi)\) corresponds to a node of the pentanomial tree.

### 4.1.2 Structure of the \((q, A)\)-grid

At each node of the tree, we compute the values of \(\Theta_n^{\xi}(q, A)\) for \((q, A) \in \mathcal{G}_q \times \mathcal{G}_A\), where \(\mathcal{G}_q\) is a grid of the form

\[
\left\{ \frac{k}{n_q - 1} \right\}_{k \in \{0, \ldots, n_q - 1\}}
\]

and where \(\mathcal{G}_A\) is a grid of the form

\[
\left\{ S_0 + \xi \left( \frac{k}{n_A - 1} - \frac{1}{2} \right) \sigma \sqrt{N \delta t} \right\}_{k \in \{0, \ldots, n_A - 1\}}.
\]

For each grid, we need to choose the number of points in the grid \((n_q, n_A)\) and the width of the grid (linked to \(q_{\text{max}}\) and \(\xi\)).

The number of shares to deliver is linked to the ratio \(F/A\). Consequently, it is unbounded and we need to choose a minimal value for \(A\). Since \(\mathbb{V}[A_N] \approx \frac{1}{2} N \sigma^2 \delta t\), it is natural to consider values of \(A\) in an interval of the type

\[
(S_0 - \frac{1}{\sqrt{3}} \omega \sigma \sqrt{N \delta t}, S_0 + \frac{1}{\sqrt{3}} \omega \sigma \sqrt{N \delta t}),
\]

where \(\omega\) is the number of standard deviations we want to keep. In practice, we choose \(\xi = \frac{2}{\sqrt{3}} \omega = 3\), that is, around 2.6 standard deviations. Then, it is natural to consider, in coherence with the lower bound for \(A\), a value

\[
q_{\text{max}} \approx \frac{F}{S_0 - \frac{1}{2} \xi \sigma \sqrt{N \delta t}}.
\]

In our numerical method, we consider the optimization of the execution strategy on the grid, that is,

\[
\tilde{\Theta}_{n,n+1}^{\xi}(q, A) = \min_{q_{\text{target}} \in I_{q,n} \cap \mathcal{G}_q} \frac{1}{\gamma} \log \left( \mathbb{E} \left[ \exp \left( -\gamma \left( q \sigma \sqrt{\delta t} \varepsilon_{n+1} - L \left( \frac{q_{\text{target}} - q}{V_{n+1} \delta t} \right) V_{n+1} \delta t \right. \right. \right. \\
+ \left. \left. \left. \left. \Theta_n^{\xi+(\varepsilon_{n+1}+2)}(q_{\text{target}}, A_{n,A}^{'(\cdot)}) \right) \right) \right) \right],
\]

where \(I_{q,n} = [q + \rho V_{n+1} \delta t, q + \bar{\rho} V_{n+1} \delta t]\).
Remark 4.1 In practice, if \((V_n)_n\) is a constant process, it is better to have \(\rho V \delta t\) and \(\hat{\rho} V \delta t\) on the \(q\)-grid.

For computing the minimum in the previous equation, and hence for computing \(\tilde{\Theta}_{n,n+1}^\xi(q,A)\), we need the values of \(\Theta_{n+1}^{\xi+(\kappa_{n+1}+2)}(q_{\text{target}}, A_{n,A}')\). As \(A_{n,A}'\) does not necessarily lie on the grid \(G_A\), we use interpolation with natural cubic splines and, if necessary, we extrapolate linearly the functions outside of the domain (for \(A\)).

### 4.1.3 Determination of the optimal strategy

Using the above recursive equations, we can approximate the values of the functions \((\theta_n)_n\). Moreover, at each time step and for each combination of the tuple \((q, S, A)\) for which we computed the value, we can easily compute the optimal strategy. We obtain \(q^*_\text{target}\) from the minimization procedure that permits us to compute \(\tilde{\Theta}_{n,n+1}^\xi(q,A)\). If

\[
\tilde{\Theta}_{n,n+1}^\xi(q,A) > F\left(\frac{S_0 + \sigma \sqrt{\delta t}(\xi - 2n)}{A} - 1\right) + \epsilon \left(\frac{F}{A} - q\right),
\]

and if we are allowed to exercise the option, then we do so. Otherwise, we send an order of size \(q^*_\text{target} - q\) to the market.

### 4.2 Examples

#### 4.2.1 Reference scenarios

We now test our numerical method on practical examples. We consider the following reference case, which corresponds to rounded values for a major stock of the CAC 40 index. This case will be used throughout the remainder of the text.

Parameters for the stock:
- \(S_0 = €45\);
- \(\sigma = 0.6\), which corresponds to an annual volatility of approximately 21%;
- \(V = 4,000,000\) stocks per day;
- \(L(\rho) = |\eta|\rho|^{1+\phi}\) with \(\eta = 0.1\) and \(\phi = 0.75\).

Parameters for the ASR contract:
- \(T = 63\) days;
- the set of possible dates for early exercising the option is \(\mathcal{N} = [22, 62] \cap \mathbb{N}\);
- \(F = €900,000,000\).
Parameters for the bank:

- participation rates are bounded by 25% and 25%;
- risk aversion is 2.5 \times 10^{-7} €^{-1};
- \ell is as described in Remark 2.2 (in other words, after the option has been exercised, we assume that execution occurs at a constant participation rate of 25%).

Parameters for the numerical method:

- \( q_{\text{max}} = 25 \, 000 \, 000 \) stocks;
- \( n_q = 201 \);
- \( \xi = 3 \), corresponding to \( \omega \approx 2.6 \) standard deviations;
- \( n_A = 21 \).

We first start with three examples corresponding to three trajectories for the price.

The first example, in Figure 1, corresponds to an increasing trend, and the stock price is therefore above its average. As explained in the previous section, there is no reason in that case to exercise the option early as \( A \) increases (the dot corresponds to the date when the option is exercised).

The second example, in Figure 2, corresponds to a stock price oscillating around its average value. We see, as expected, that the buying process accelerates when the stock price decreases, and decelerates when the stock price increases. We even see that the buying process can turn into a selling process as prices increase. The rationale for this is hedging, and it can be seen from the term we called “risk term I”:

\[
\sum_{j=n}^{n^* - 1} \left( q_j - \frac{j}{n^* A_{n^*}} \right) \varepsilon_{j+1}.
\]

As prices increase, we expect the eventual number of shares to be bought (that is, \( F/A_{n^*} \)) to decrease and the exercise date (ie, \( t_{n^*} \)) to be later than initially thought. Therefore, the term

\[
\frac{j}{n^* A_{n^*}} \frac{F}{A_{n^*}}
\]

decreases, and \( q \) has to decrease in order to reach the same level of risk. We also see in this second example that the option is exercised near maturity.

As the optimal strategy leads to buying and selling shares, it is interesting to understand what happens when we force the bank to use a buy-only strategy. If we impose \( \rho = 0 \), we obtain the optimal strategy of Figure 3, which is substantially different.
The third trajectory we consider corresponds to decreasing prices (see Figure 4). In this case, the bank buys shares quite rapidly and exercises the option as soon as it can, although it has not yet bought the required number of shares. By exercising the option, the bank in fact wants to avoid the natural decrease in $A$ that would lead to an increase in the number of shares it would have to buy.

In addition to optimal strategies, we can compute, in our reference case, the price $\Pi$ of the fixed-notional ASR contract. Here, this price is negative, as we
found $\Pi = -10,669,023 \simeq -1.185\% F$. This means that the gain associated with the optionality component of the ASR contract is important enough to compensate (in utility terms) the execution costs and the risk of the contract. When we impose the use of a buy-only strategy, we obtain $\Pi = -10,330,135 \simeq -1.148\% F$.

### 4.2.2 Comparative statics

In order to understand the role of the main parameters, we carry out comparative statics. We focus on the liquidity of the stock, through the parameter $\eta$, on the volatility of the stock price, through $\sigma$, and on the risk-aversion parameter, $\gamma$.

**Effect of execution costs**

Regarding execution costs, we consider our reference case with three values for the parameter $\eta$: 0.01, 0.1 and 0.2. We concentrate on the second price trajectory (Figure 5), as it shows the role of $\eta$ very well: the more liquid the stock, the larger the number of round trips on the stock for hedging purposes.

The differences in terms of prices are given in Table 1. As expected, the more liquid the stock, the lower the indifference price: the buying process is indeed less costly in itself, and round trips on the stock to hedge risk are also less costly.
FIGURE 4  Optimal strategy when prices are mainly going down.

![Graph showing optimal strategy for decreasing prices.](image)

FIGURE 5  Optimal strategy when prices are mainly oscillating, for different values of $\eta$.

![Graph showing optimal strategy for oscillating prices.](image)

TABLE 1  Price of the ASR contract for different values of the liquidity parameter $\eta$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$\Pi/F$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>-1.254</td>
</tr>
<tr>
<td>0.1</td>
<td>-1.185</td>
</tr>
<tr>
<td>0.2</td>
<td>-1.117</td>
</tr>
</tbody>
</table>
TABLE 2  Price of the ASR contract for different values of the volatility parameter $\sigma$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\Pi/F$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>-2.163</td>
</tr>
<tr>
<td>0.6</td>
<td>-1.185</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.605</td>
</tr>
</tbody>
</table>

Effect of volatility

We now consider our reference case with three values for the volatility parameter $\sigma$: 0.3, 0.6 and 1.2. There are two effects. On the one hand, the more volatile the stock, the more important the magnitude of the terms we called “risk term I” and “post-exercise risk-liquidity term”. On the other hand, the larger the $\sigma$, the more valuable the option for the bank.

In terms of prices, we see in Table 2 that the first effect dominates, as $\Pi$ is an increasing function of $\sigma$. We also see that $\sigma$ is a very important driver of the price of an ASR contract.

Effect of risk aversion

We go on with risk aversion. We consider our reference case with four values for the parameter $\gamma$: 0, $2.5 \times 10^{-9}$, $2.5 \times 10^{-7}$ and $2.5 \times 10^{-6}$. Figure 6 shows the complex influence of $\gamma$ on the optimal strategy for the second stock price trajectory where prices mainly oscillate.

The first thing we see is that when $\gamma = 0$ the bank buys almost the same number of shares every day, the changes being due to changes in the targeted number of shares to be bought (as $A$ oscillates).

In the case of a risk-averse agent, we see different behaviors depending on the level of risk aversion, because there are several effects at work. The randomness indeed occurs in both the execution process and the number of shares to buy. On the one hand, risk aversion encourages us to exercise the option as soon as $S$ is below $A$ to avoid the first effect of risk. However, exercising early reduces the period during which the former risk is partially hedged. The type of risk that matters more depends on the value of $\gamma$. This is why we observe so much variety in the optimal strategy. Choosing the value of $\gamma$ corresponding to the risk the bank wants to bear is therefore very important.15

14 We can extend the results to the case $\gamma = 0$.

15 This is a classical problem in models involving an expected utility criterion. It is very often encountered in optimal execution (for instance, in models of Almgren–Chriss type) and in asset management (for instance, in the Markowitz approach, or in the Merton model).
FIGURE 6  Optimal strategy when prices are mainly oscillating, for different values of $\gamma$.

TABLE 3  Price of the ASR contract for different values of the risk aversion parameter $\gamma$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\Pi / F$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1.499</td>
</tr>
<tr>
<td>$2.5 \times 10^{-9}$</td>
<td>-1.490</td>
</tr>
<tr>
<td>$2.5 \times 10^{-7}$</td>
<td>-1.185</td>
</tr>
<tr>
<td>$2.5 \times 10^{-6}$</td>
<td>-0.468</td>
</tr>
</tbody>
</table>

What is very clear, however, is the influence of $\gamma$ on the price of the ASR contract: the more risk averse the bank, the less it values signing an ASR contract with a firm (Table 3).

5 CONCLUSION

This paper is a contribution to a new literature on option pricing and hedging, inspired by that on optimal execution. Specifically, we have presented a model for characterizing and computing the optimal strategy of a bank entering a fixed-notional ASR contract with a firm. Our discrete-time model enables us to model the interactions between the execution problem linked to an ASR and the Asian/Bermudan option that is part of the contract, whereas classical approaches would separate the two. In addition to providing an optimal strategy, we define an indifference price for the contract.
The numerical method we developed for approximating solutions works very well in practice and provides both optimal strategies and prices.

DECLARATION OF INTEREST

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