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Adapting the Basel II advanced internal-ratings-based models for International Financial Reporting Standard 9
Peter Miu and Bogie Ozdemir
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CONTENTS

RESEARCH PAPERS

*Portfolio credit risk model with extremal dependence of defaults and random recovery*
Jong-June Jeon, Sunggon Kim and Yonghee Lee
1

*Primary-firm-driven portfolio loss*
Stuart M. Turnbull
33

*Adapting the Basel II advanced internal-ratings-based models for International Financial Reporting Standard 9*
Peter Miu and Bogie Ozdemir
53
Research Paper

Portfolio credit risk model with extremal dependence of defaults and random recovery

Jong-June Jeon, Sunggon Kim and Yonghee Lee

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ABSTRACT

The extremal dependence of defaults, and negative correlation between defaults and their recovery rates, are of major interest in modeling portfolio credit risk. In order to incorporate these two features, we propose a portfolio credit risk model with random recovery rates. The proposed model is an extension of the traditional $t$-copula model for the credit portfolio with constant recovery rates. A skew-normal copula model is adopted to represent dependent random recovery rates. In our proposed model, various types of dependency between the defaults and their recovery rates are possible, including an inverse relation. We also propose a conditional Monte Carlo simulation algorithm for estimating the probability of a large loss in the model, and an importance sampling version of it. We show that the proposed Monte Carlo simulation algorithm is relatively efficient compared with the plain Monte Carlo simulation. Numerical results are presented to show the performance and efficiency of the algorithms.

Keywords: portfolio credit risk; random recovery; extreme loss probability; importance sampling; conditional Monte Carlo simulation.

1 INTRODUCTION

An important problem in risk management is the modeling of rare events related to portfolio loss. The amount of portfolio loss is determined by the sum of individual
losses, which are represented by two components: obligors’ defaults and recovery rates. Defaults are correlated by trading and transferring credit-related risks. Typically, an obligor’s default is modeled by a binary random variable that is a function of latent random variables indicating the credit risk of the obligor. A copula function provides a useful framework to model the dependence structure by constructing a multivariate distribution of the latent random variables (Frey and McNeil 2003; Li 2000). For example, Glasserman et al (2002) successfully captured the extremal dependence of obligor defaults.

As another component of individual losses, the recovery rate of a defaulted obligor plays an important role in evaluating portfolio loss distribution. The recovery rate is the proportion of a bad debt that can be recovered in the event of default. The higher the recovery rate of the obligor, the smaller the loss due to obligor default. Hence, the individual recovery rate affects the realized total loss of a portfolio. Glasserman and Li (2005), Bassamboo et al (2008) and Chan and Kroese (2010) assume that the recovery rates are constant and known, while Andersen and Sidenius (2004) assume that the recovery rates are dependent random variables.

It is crucial to develop a method for estimating the portfolio loss probability as well as for model building. Due to the interdependence of obligors’ defaults and recovery rates, it is barely possible to represent the distribution of the total loss in a closed form, and usually the probability of a large loss is estimated by Monte Carlo simulation. However, if a large loss of interest rarely occurs, a crude Monte Carlo simulation is not efficient nor reliable. Thus, an efficient sampling algorithm needs to be developed. Glasserman and Li (2005) proposed an importance sampling algorithm for a portfolio credit risk model derived from the normal copula, and showed the optimality of the algorithm. Bassamboo et al (2008) extended the portfolio credit risk model by adopting the $t$-copula function, and showed the sharp asymptotics for the loss probability. Chan and Kroese (2010, 2011, 2012) developed a conditional Monte Carlo algorithm with the $t$-copula function, which is simple but very efficient, and proved the relative efficiency of the algorithm.

Most of the algorithms developed are restricted to the portfolio credit risk model with fixed recovery rates (see, for example, Bassamboo et al 2008; Chan and Kroese 2010; Glasserman and Li 2005). However, there are many studies giving empirical evidence that recovery rates tend to be inversely related to default rates (see, for example, Acharya et al 2003; Altman et al 2005; Frey and McNeil 2003; Perraudin and Hu 2006). These suggest that portfolio credit risk models with constant recovery rates are inadequate in many situations. Even though Andersen and Sidenius (2004) proposed a portfolio credit risk model with dependent defaults and random recovery rates, their model is based on the normal copula function, so it has limited power to explain the extremal dependence of defaults.
In this paper, we propose a portfolio credit risk model with extremal dependence of defaults and dependent random recovery rates. In our proposed model, the default probability of an obligor is inversely related to the recovery rate at default. Our model is a combined extension of the models by Bassamboo et al. (2008) and Andersen and Sidenius (2004). We propose a factor copula model to represent dependence in default events and to represent random recovery rates. Our model incorporates the two dependence structures simultaneously. Advanced models, including those of Andersen and Sidenius (2004) and Bassamboo et al. (2008), are covered by the proposed model. Also, we investigate the dependence structure of the proposed model via Kendall’s tau measure, extremal tail dependence, and show the sharp asymptotics of the large loss probability. To estimate the large loss probability in the proposed model, we propose a conditional Monte Carlo algorithm, which is an extension of Chan and Kroese (2010). Further, we prove the relative efficiency of our proposed algorithm and confirm its efficiency by simulation results.

This paper is organized as follows. Section 2 explains the proposed model for portfolio credit risk with random recovery. Section 3 proposes a conditional Monte Carlo simulation algorithm and its extension. Section 4 shows the relative efficiency of the proposed algorithm. Simulation results are included in Section 5, and calibration of the model parameters is also treated in this section. Concluding remarks follow in Section 6.

2 THE PORTFOLIO STRUCTURE AND DEPENDENCE

We consider a portfolio of loans consisting of \( n \) obligors. Over a fixed time horizon, each obligor can default, and the amount of loss due to the default of an obligor is assumed to be random. Let \( p_i, i = 1, 2, \ldots, n \), be the default probability of obligor \( i \). We assume that there are latent variables \( X_1, \ldots, X_n \) such that the default of obligor \( i \) occurs if \( X_i \) exceeds a given threshold \( x_i, i = 1, 2, \ldots, n \). The value of \( x_i \) is chosen to match the default probability \( p_i \), ie,

\[
\Pr\{X_i > x_i\} = p_i, \quad i = 1, \ldots, n.
\]

Let \( c_i > 0, i = 1, 2, \ldots, n \), be the maximum loss due to the default of obligor \( i \), and let \( R_i, 0 \leq R_i \leq 1, i = 1, 2, \ldots, n \), be the recovery rate of the \( i \)th obligor. Then, the realized loss at the default of obligor \( i \) is \( c_i (1 - R_i) \). The total loss of the portfolio is given by

\[
L_n = \sum_{i=1}^{n} c_i (1 - R_i) \mathbb{I}(X_i > x_i), \tag{2.1}
\]

where \( \mathbb{I}(A) \) is the indicator function of event \( A \).
The traditional method for modeling the dependence between obligors introduces a number of common risk and shock factors into \( X_1, \ldots, X_n \) and \( R_1, \ldots, R_n \). These factors affect the risk levels of all obligors. The default event and the recovery rate may have different factors. In this paper, we consider the simplest case of a common risk factor \( Z \) shared by recovery rates and default events, and a shock factor \( \lambda \) corresponding only to the default events.

There is significant empirical evidence by Oh and Patton (2017) that the common risk factor is asymmetrically distributed. In order to introduce asymmetry into the distribution of \( Z \), we assume that \( Z \sim SN(0, 1, \alpha) \), the skew-normal distribution with location parameter 0, scale parameter 1 and shape parameter \( \alpha \) \((-\infty < \alpha < \infty)\). Then, the probability density function (pdf) of \( Z \) is given by

\[
f_{SN}(z; 0, 1, \alpha) = 2\phi(z)\Phi(\alpha z), \quad -\infty < z < \infty, \tag{2.2}\]

where \( \phi(z) \) and \( \Phi(z) \) denote the pdf and the cumulative distribution function (cdf) of the standard normal random variable, respectively.

For modeling the dependent defaults of the obligors, we propose a multivariate skew \( t \)-copula model for \( X = (X_1, X_2, \ldots, X_n) \) given by

\[
X_i = \frac{\rho_i Z + \sqrt{1 - \rho_i^2} \eta_i}{\sqrt{\lambda}}, \quad i = 1, 2, \ldots, n, \tag{2.3}\]

where \( \rho_i \) is a constant in \((0, 1)\), \( \eta_i \sim N(0, 1) \) and \( \lambda \sim Gamma(\nu/2, 2/\nu) \), ie, the gamma distribution with shape parameter \( \nu/2 \) and scale parameter \( 2/\nu \) \((\nu > 0)\). In the above equation, \( \eta_i \) denotes the idiosyncratic risk factor associated with obligor \( i \), \( i = 1, 2, \ldots, n \). To model the positive and nontrivial correlation between \( Z \) and \( X_i \), we assume that \( \rho_i \in (0, 1) \) for \( i = 1, 2, \ldots, n \).

For modeling the dependent recovery rates of the obligors, we propose that the recovery rate of obligor \( i \), \( i = 1, 2, \ldots, n \), is determined by the common risk factor \( Z \) and the idiosyncratic risk factor \( \xi_i \) as follows:

\[
R_i = G(\mu_i - (\tilde{\rho}_i Z + \xi_i)), \quad i = 1, 2, \ldots, n, \tag{2.4}\]

where \( G(x) \) is a strictly increasing function, with \( G(-\infty) = 0 \) and \( G(\infty) = 1 \); \( \mu_i \) is a constant in \((-\infty, \infty)\); \( \tilde{\rho}_i \) is a constant in \((0, \infty)\); and \( \xi_i \sim N(0, \sigma^2) \). We assume that all \( Z, \lambda \) and \( \eta_i, \xi_i \) \((i = 1, \ldots, n)\) are mutually independent. It is clear from (2.3) and (2.4) that the default probability and the recovery rate of an obligor are inversely related.

By Azzalini (2013, p. 26), \( \rho_i Z + \sqrt{1 - \rho_i^2} \eta_i \), the numerator of \( X_i \) in (2.3), also follows the skew-normal distribution \( SN(0, 1, \alpha_i) \) for \( i = 1, 2, \ldots, n \), where

\[
\alpha_i = \frac{\alpha}{\sqrt{1 + (1 + \alpha^2)(1 - \rho_i^2)/\rho_i^2}}. \]
Then, $X_i, i = 1, 2, \ldots, n,$ follows the skew $t$ distribution $\text{ST}(0, 1, \alpha_i, \nu)$ with location parameter 0, scale parameter 1, shape parameter $\alpha_i$ and degree of freedom $\nu$ (Azzalini 2013, p. 102). The pdf of $X_i$ is given by

$$t(x; \alpha_i, \nu) = 2t(x; \nu)T \left( \alpha_i x \sqrt{\frac{\nu + 1}{\nu + x^2}} ; \nu + 1 \right),$$

(2.5)

where $t(x; \nu)$ and $T(x; \nu)$ denote the pdf and the cdf of the Student $t$ random variable with degree of freedom $\nu$. The skewness of $X_i$ is controlled by the value of $\alpha$.

Azzalini and Valle (1996) defined an $n$-dimensional random vector $Y$, which follows the $n$-dimensional multivariate skew-normal distribution with location parameter 0, correlation matrix $\Omega$ and shape parameter $\alpha$ if and only if the pdf of $Y$ is given by

$$f(y) = 2\varphi_n(y; \Omega)\Phi(\alpha^T y), \quad y \in \mathbb{R}^n,$$

where $\varphi_n(y; \Omega)$ denotes the pdf of the $n$-dimensional standard normal vector with positive definite $n \times n$ correlation matrix $\Omega$, and $\alpha \in \mathbb{R}^n$. We define $Y \sim \text{SN}_n(0, \Omega, \alpha)$. By Azzalini and Capitanio (2003), the random vector $Y/\sqrt{\lambda}$ follows the $n$-dimensional multivariate skew $t$ distribution $\text{ST}_n(0, \Omega, \alpha, \nu)$ if $\lambda \sim \text{Gamma}(v/2, 2/\nu) (\nu > 0)$ and is independent of $Y$. The pdf of $Y/\sqrt{\lambda}$ is given by

$$f(x) = 2t_n(x; \Omega, \nu)T \left( \alpha^T x \sqrt{\frac{\nu + n}{\nu + Q(x)}} ; \nu + n \right), \quad x \in \mathbb{R}^n,$$

(2.6)

where $t_n(x; \Omega, \nu)$ is the pdf of the $n$-dimensional Student $t$ distribution with correlation matrix $\Omega$ and degree of freedom $\nu$, and $Q(x) = x^T \Omega^{-1} x$.

**Theorem 2.1** The random vector $X = (X_1, X_2, \ldots, X_n)$ follows the multivariate skew $t$ distribution $\text{ST}_n(0, \Omega, \tilde{\alpha}, \nu)$, where $\Omega$ is an $n \times n$ matrix with diagonal elements 1 and off-diagonal $(i, j)$ elements $\rho_i \rho_j$, ie,

$$\Omega = \begin{bmatrix}
1 & \rho_1 \rho_2 & \cdots & \rho_1 \rho_n \\
\rho_2 \rho_1 & 1 & \cdots & \rho_2 \rho_n \\
\vdots & \vdots & \ddots & \vdots \\
\rho_n \rho_1 & \rho_n \rho_2 & \cdots & 1
\end{bmatrix},$$

and

$$\tilde{\alpha} = \delta(1 - \delta^2 \rho^T \Omega^{-1} \rho)^{-1/2} \Omega^{-1} \rho$$

with $\delta = \alpha/\sqrt{1 + \alpha^2}$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_n)^T$.

**Proof** Let $Y = (Y_1, Y_2, \ldots, Y_n)^T$, where

$$Y_i = \rho_i Z + \sqrt{1 - \rho_i^2} \eta_i, \quad i = 1, 2, \ldots, n.$$  

(2.7)
Then, $X = Y / \sqrt{\lambda}$, which means that it is sufficient to show $Y \sim \text{SN}(0, \Omega, \tilde{\alpha})$. To this end, we derive $M_Y(t)$, the moment generating function (MGF) of $Y$, where $t = (t_1, t_2, \ldots, t_n)^T$. From (2.7), we have

$$M_Y(t) = E \left[ \exp \left\{ t^T \rho Z + \sum_{i=1}^{n} t_i \sqrt{1 - \rho_i^2} \eta_i \right\} \right].$$

Since the MGF of $Z$ is given by

$$M_Z(t) = 2 \exp \{ \frac{1}{2} t^2 \} \Phi(\delta t),$$

and that of $\eta_i$ is equal to $\exp \{ \frac{1}{2} t^2 \}$, we have that

$$M_Y(t) = 2 \exp \left\{ \frac{(t^T \rho)^2}{2} + \frac{1}{2} \sum_{i=1}^{n} (1 - \rho_i^2) t_i^2 \right\} \Phi(\delta t^T \rho)$$

$$= 2 \exp \{ \frac{1}{2} t^T \Omega t \} \Phi(\delta \rho^T t). \quad (2.8)$$

Due to Azzalini and Valle (1996), the above form of the MGF of $Y$ is equal to that of $\text{SN}(0, \Omega, \tilde{\alpha})$, where $\tilde{\alpha}$ is the unique solution of the following equation:

$$\delta \rho = \frac{\Omega \tilde{\alpha}}{(1 + \tilde{\alpha}^T \Omega \tilde{\alpha})^{1/2}}.$$  

The solution of the above equation is given by

$$\tilde{\alpha} = \frac{\delta \Omega^{-1} \rho}{(1 - \delta^2 \rho^T \Omega^{-1} \rho)^{1/2}},$$

which completes the proof. $\square$

The joint distribution of $X_i$ and $X_j$ can be obtained by restricting $X$ on $(X_i, X_j)$ in the above theorem. We define

$$\Omega_{ij} = \begin{bmatrix} 1 & \rho_i \rho_j \\ \rho_i \rho_j & 1 \end{bmatrix},$$

and

$$\alpha_{ij} = \delta(1 - \delta^2 (\rho_i, \rho_j) \Omega_{ij}^{-1} (\rho_i, \rho_j)^T)^{-1/2} \Omega_{ij}^{-1} (\rho_i, \rho_j)^T$$

$$= \alpha^2 \sqrt{ (1 - \rho_i^2 \rho_j^2) (\alpha^2 (1 - \rho_i^2) (1 - \rho_j^2) + (1 - \rho_i^2 \rho_j^2) ) } \begin{bmatrix} \rho_i (1 - \rho_j^2) \\ \rho_j (1 - \rho_i^2) \end{bmatrix}. \quad (2.9)$$

Then, Theorem 2.1 says that the random vector $(X_i, X_j)$ follows the two-dimensional skew $t$ distribution $\text{ST}_2(0, \Omega_{ij}, \alpha_{ij}, v)$ for $1 \leq i, j \leq n, i \neq j$. 

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Our proposed model for $X_1, X_2, \ldots, X_n$ is an extension of the $t$-copula model analyzed by Bassamboo et al (2008). The $t$-copula model has the same form as (2.3), except that $Z$ follows the standard normal distribution. In the model, $X_i$ follows the Student $t$ distribution, which is symmetric. However, in some circumstances, especially when financial crises spread and the defaults of obligors are contagious, it is not appropriate to assume that $X_i, i = 1, 2, \ldots, n$, is symmetric. In our proposed model, the asymmetry of $X_1, X_2, \ldots, X_n$ is controlled by the parameter $\alpha$. When $\alpha = 0$, our model is equal to the $t$-copula model. For positive values of $\alpha$, the distribution of $X_i, i = 1, 2, \ldots, n$, is right skewed, which is an appropriate type of distribution for the random variable representing the riskiness of an obligor.

From (2.3) and (2.4), we can see that the dependence between $X_1, \ldots, X_n$ is determined by $\rho_i, i = 1, 2, \ldots, n$, and that the dependence between $R_1, \ldots, R_n$ is determined by $\sigma \xi$ and $\tilde{\rho}_i, i = 1, 2, \ldots, n$. The following theorem states the strength of the dependence in terms of Kendall’s tau correlations.

**Theorem 2.2**  The Kendall’s tau correlations of $X_1, \ldots, X_n$ for $1 \leq i, j \leq n$, $i \neq j$ are given by

$$
\tau(X_i, X_j) = 4 \int_{\mathbb{R}^2} F_v \left( \frac{\rho_{ij} x}{\sqrt{1 - \rho_{ij}^2}} \right) F_v \left( \frac{\rho_{ij} y}{\sqrt{1 - \rho_{ij}^2}} \right) t(y; \alpha, \nu) t(x + y; \alpha, \nu) \, dy \, dx - 1,
$$

where $F_v(x)$ is the cdf of the difference of two independent Student $t$ random variables with degree of freedom $\nu$. The Kendall’s tau correlations of $R_1, \ldots, R_n$ for $1 \leq i, j \leq n$, $i \neq j$ are given by

$$
\tau(R_i, R_j) = 4 \int_{\mathbb{R}^2} \Phi \left( \frac{\tilde{\rho}_{ij} x}{\sqrt{2\sigma \xi}} \right) \Phi \left( \frac{\tilde{\rho}_{ij} y}{\sqrt{2\sigma \xi}} \right) f_{SN}(y; \alpha) f_{SN}(x + y; \alpha) \, dy \, dx - 1.
$$

**Proof** Let $(Z', \lambda', \eta'_1, \ldots, \eta'_n, \xi'_1, \ldots, \xi'_n)$ be an independent random copy of $(Z, \lambda, \eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n)$, and let $X'_i, 1 \leq i \leq n$, be an independent random copy of $X_i$, ie,

$$
X'_i = \frac{\rho_i Z' + \sqrt{1 - \rho_i^2} \eta'_i}{\sqrt{\lambda'}}.
$$

The Kendall’s tau correlation of $X_i$ and $X_j$ is defined by

$$
\tau(X_i, X_j) = \Pr\{(X_i - X'_i)(X_j - X'_j) > 0\} - \Pr\{(X_i - X'_i)(X_j - X'_j) < 0\}.
$$

Then, we have

$$
\tau(X_i, X_j) = 2 \Pr\{(X_i - X'_i)(X_j - X'_j) > 0\} - 1 \Rightarrow 2 \Pr\{X_i - X'_i > 0, X_j - X'_j > 0\} \quad + \Pr\{X_i - X'_i < 0, X_j - X'_j < 0\}) - 1.
$$
Due to the symmetry of the pdf of \((X_i - X'_i, X_j - X'_j)\) about the origin, we have

\[
\tau(X_i, X_j) = 4 \Pr\{X_i - X'_i > 0, X_j - X'_j > 0\} - 1. \quad (2.10)
\]

From the definitions of \(X_i, X'_i, X_j,\) and \(X'_j,\) we derive

\[
\Pr\{X_i - X'_i > 0, X_j - X'_j > 0\} = \Pr\left\{ \frac{\rho_i}{\sqrt{1 - \rho_i^2}} \tilde{Z} - U_i > 0, \frac{\rho_j}{\sqrt{1 - \rho_j^2}} \tilde{Z} - U_j > 0 \right\},
\]

where

\[
\tilde{Z} = \frac{Z}{\sqrt{\lambda}} - \frac{Z'}{\sqrt{\lambda'}},
\]

\[
U_i = \frac{\eta_i}{\sqrt{\lambda}} - \frac{\eta_i}{\sqrt{\lambda'}},
\]

\[
U_j = \frac{\eta_j}{\sqrt{\lambda}} - \frac{\eta_j}{\sqrt{\lambda'}}.
\]

Let \(F_v(x)\) be the cdf of the difference of two independent Student \(t\)-random variables with degree of freedom \(v\). Then,

\[
F_v(x) = \int_{-\infty}^{\infty} t(y; v) T(x + y; v) \, dy, \quad x \in \mathbb{R}.
\]

Clearly, \(U_i\) and \(U_j\) are independent and identically distributed with cdf \(F_v(x)\), which implies that

\[
\Pr\{X_i - X'_i > 0, X_j - X'_j > 0\} = E_{\tilde{Z}} \left[ F_v\left( \frac{\rho_i}{\sqrt{1 - \rho_i^2}} \tilde{Z} \right) F_v\left( \frac{\rho_j}{\sqrt{1 - \rho_j^2}} \tilde{Z} \right) \right]. \quad (2.11)
\]

The random variables \(Z/\sqrt{\lambda}\) and \(Z'/\sqrt{\lambda'}\) are independent and have the same distribution, \(ST(0, 1, \alpha, v)\), with the pdf given by (2.5). It can be easily shown that the pdf of \(\tilde{Z}\) is given by

\[
f_{\tilde{Z}}(x; \alpha, v) = \int_{-\infty}^{\infty} t(y; \alpha, v)t(x + y; \alpha, v) \, dy.
\]

Applying the above form of the pdf of \(\tilde{Z}\) to (2.11) completes the first part of the proof.

In the same manner as that above, we obtain

\[
\tau(R_i, R_j) = 4 \Pr\{R_i - R'_i > 0, R_j - R'_j > 0\} - 1.
\]

Since \(G(x)\) in (2.4) is strictly increasing, it follows that

\[
\tau(R_i, R_j) = 4 \Pr\{\xi_i - \xi'_i < \tilde{\rho}_i (Z' - Z), \xi_j - \xi'_j < \tilde{\rho}_j (Z' - Z)\} - 1.
\]
Since $(\xi_i - \xi_j')/\sqrt{2\sigma_\xi}$ and $(\xi_j - \xi_j')/\sqrt{2\sigma_\xi}$ are independent and have the standard normal distribution, we have

$$\tau(R_i, R_j) = 4E \left[ \Phi \left( \frac{\tilde{\rho}_i}{\sqrt{2}\sigma_\xi} (Z' - Z) \right) \Phi \left( \frac{\tilde{\rho}_j}{\sqrt{2}\sigma_\xi} (Z' - Z) \right) \right] - 1. \quad (2.12)$$

The random variables $Z$ and $Z'$ are independent and have the same density, $f_{SN}(0, 1, \alpha)$, given by (2.2). Then, the pdf of $Z - Z'$ can be derived as

$$f_{Z - Z'}(x; \alpha) = \int_{-\infty}^{\infty} f_{SN}(y; \alpha) f_{SN}(x + y; \alpha) \, dy.$$  

Applying the above form of the pdf of $Z - Z'$ to (2.12) completes the second part of the proof. \hfill \square

From the above theorem, we can see that $\tau(R_i, R_j)$ does not depend on the values of $\mu_i$ and $\mu_j$. Figure 1 shows Kendall’s tau correlations of $X_i$ and $X_j$ when $\rho_i = \rho_j = \rho$, and those of $R_i$ and $R_j$ when $\tilde{\rho}_i = \tilde{\rho}_j = \tilde{\rho}$. We consider the cases with $\nu = 4$ and various values of $\alpha$, $\rho$ and $\tilde{\rho}$. As expected, the Kendall’s tau correlation of $X_i$ and $X_j$ increases as $\rho$ increases, while the correlation decreases as the value of $\alpha$ increases. The Kendall’s tau correlation of recovery rates shows the same behavior.

In particular, when $\alpha = 0$, the random variables $X_1, X_2, \ldots, X_n$ are identically distributed with the Student $t$ of degree of freedom $\nu$, and $\tilde{\rho}_i Z + \xi_i \sim N(0, \tilde{\sigma}_i^2 + \sigma_\xi^2)$, $i = 1, 2, \ldots, n$. In this case, the Kendall’s $\tau$ correlations between $X_i$ and $X_j$ for $i, j = 1, 2, \ldots, n$, and between $R_i$ and $R_j$ for $i, j = 1, 2, \ldots, n$, have a simpler form compared with the $\alpha \neq 0$ case.

**Theorem 2.3** Suppose that $\alpha = 0$, ie, the common risk factor $Z$ follows the standard normal distribution. Then, the Kendall’s tau correlations of $X_1, \ldots, X_n$ and $R_1, \ldots, R_n$ are given, for $1 \leq i, j \leq n$, $i \neq j$, by

$$\tau(X_i, X_j) = \frac{2}{\pi} \arcsin(\rho_i \rho_j),$$

$$\tau(R_i, R_j) = \frac{2}{\pi} \arcsin \left( \frac{\tilde{\rho}_i \tilde{\rho}_j}{\tilde{\sigma}_i \tilde{\sigma}_j} \right),$$

where $\tilde{\sigma}_i^2 = \tilde{\rho}_i^2 + \sigma_\xi^2$.

**Proof** We use the same notation as in the proof of Theorem 2.2. By conditioning on the values of $\lambda$ and $\lambda'$, it follows from (2.10) that

$$\tau(X_i, X_j) = 4E_{\lambda, \lambda'} [\Pr\{X_i - X_i' > 0, X_j - X_j' > 0 \mid \lambda, \lambda']]] - 1. \quad (2.13)$$

For fixed $\lambda, \lambda' > 0$ and $i \neq j$, the random vector $(X_i - X_i', X_j - X_j')$ follows the bivariate normal distribution whose means are zeros, variances are $1/\lambda + 1/\lambda'$ and $1/\lambda + 1/\lambda'$, respectively, and the correlation is $\rho_i \rho_j$.  

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FIGURE 1  Kendall’s tau correlations for the latent variables and for the recovery rates with various values of parameters.

Note that, when $\lambda$ and $\lambda'$ are given, $\Pr\{X_i - X'_i > 0, X_j - X'_j > 0 \mid \lambda, \lambda'\}$ does not depend on the variances of $X_i - X'_i$ and $X_j - X'_j$, and thus it does not depend on the values of $\lambda$ and $\lambda'$. Due to Lindskog (2000), $\Pr\{X_i - X'_i > 0, X_j - X'_j > 0 \mid \lambda, \lambda'\} = \arcsin(\rho_i \rho_j) / 2\pi + 1/4$. By applying this result to (2.13), we obtain

$$\tau(X_i, X_j) = \frac{2}{\pi} \arcsin(\rho_i \rho_j).$$
Kendall’s tau correlation is invariant under strictly increasing transforms. From (2.4), we can see that $R_i, i = 1, 2, \ldots, n$, is a strictly increasing function of $-(\tilde{\rho}_i Z + \xi_i)$. Then,

$$
\tau(R_i, R_j) = \tau(-(\tilde{\rho}_i Z + \xi_i), -(\tilde{\rho}_j Z + \xi_j)).
$$

In a manner similar to the above, it can be shown that

$$
\tau(R_i, R_j) = \frac{2}{\pi} \arcsin \left( \frac{\tilde{\rho}_i \tilde{\rho}_j}{\tilde{\sigma}_i \tilde{\sigma}_j} \right),
$$

which completes the proof.
From Theorems 2.2 and 2.3, we can see that the proposed model can explain various dependence structures. Interestingly, when \( \alpha = 0 \), the correlation structures between recovery rates as well as between defaults, in the sense of Kendall’s tau, are the same as those of Andersen and Sidenius (2004).

When \( G \) is the standard logistic function and \( \alpha = 0 \), the distribution of random recovery rates \( R_i, i = 1, 2, \ldots n \), is known as the logit-normal distribution (Atchison and Sheng 1980) with parameters \( \mu_i \) and \( \sigma_i^2 = \hat{\rho}_i^2 + \sigma_i^2 \). Figure 2 shows the pdfs of \( R_i \) with various values of \( \mu_i \) and \( \hat{\sigma}_i \). In this case, \( \mu_i \) and \( \hat{\sigma}_i \) are estimated by the maximum likelihood method or the moment method. If \( G \) is another known function, we also estimate the values of \( \mu_i \) and \( \sigma_i^2 \) in the same way.

The values of \( \sigma_i^2 \) and \( \hat{\sigma}_i \) cannot be estimated directly from the observed values of \( R_1, R_2, \ldots, R_n \). Instead, the parameters \( (\rho_i, \hat{\rho}_i), i = 1, \ldots, n \), and \( \sigma_i^2 \) are estimated by the observed correlations according to Theorem 2.2. The calibration of the model parameters is treated in Section 5.

The comovement of the \( X_i \) and \( X_j \), \( 1 \leq i, j \leq n, i \neq j \), can be measured by the coefficients of the upper and lower tail dependence of \( X_i \) and \( X_j \), which are defined, respectively, by

\[
d_U(i, j) = \lim_{u \to 1} \Pr\{X_i > F_i^{-1}(u) \mid X_j > F_j^{-1}(u)\},
\]

\[
d_L(i, j) = \lim_{u \to 0} \Pr\{X_i \leq F_i^{-1}(u) \mid X_j \leq F_j^{-1}(u)\}.
\]

where \( F_i(x), i = 1, 2, \ldots, n \), is the cdf of \( \text{ST}(0, 1, \alpha_i, \nu) \), the distribution of \( X_i \). We are interested in the upper tail dependence, since the value of \( d_U(i, j) \) determines the probabilistic behavior of the simultaneous defaults of the obligors \( i \) and \( j \) in the event of a credit crisis. Under a slightly different representation from our definition of the multivariate skew \( t \) distribution given in (2.6), Fung and Seneta (2010) and Padoan (2011) independently obtained the coefficients of the upper and lower tail dependence. Azzalini (2013, p. 193) has rewritten the coefficient of the upper tail dependence under the same representation as ours. Applying this form of the coefficient of the upper tail dependence to our proposed model, we obtain the following theorem.

**Theorem 2.4** The coefficient of the upper tail dependence of \( X_1, \ldots, X_n \) is given for \( 1 \leq i, j \leq n, i \neq j \) by

\[
d_U(i, j) = 2 - G \left( \frac{(a_{ij}^{1/\nu} - \rho_i \rho_j) \sqrt{v + 1}}{\sqrt{1 - \rho_i^2 \rho_j^2}}; \beta_{ij}, \tau_{ij}, \nu \right) - G \left( \frac{(a_{ji}^{1/\nu} - \rho_i \rho_j) \sqrt{v + 1}}{\sqrt{1 - \rho_i^2 \rho_j^2}}; \beta_{ji}, \tau_{ji}, \nu \right),
\]
where

\[
G(x; \beta, \tau, v) = \int_{-\infty}^{x} \frac{1}{T(\tau; v)} t(z; v) T\left((\tau \sqrt{1 + \beta^2} + \beta z) \sqrt{\frac{v + 1}{v + z^2}}; v + 1\right) \, dz,
\]

\[
a_{ij} = \frac{T(\tau_{ij}; v + 1)}{T(\tau_{ij}; v + 1)}, \quad \tau_{ij} = (\rho_i \rho_j \alpha_{ij}[1] + \alpha_{ij}[2]) \sqrt{\frac{v + 1}{1 + \beta_{ij}^2}},
\]

\[
\beta_{ij} = \alpha_{ij}[1] \sqrt{1 - \rho_i^2 \rho_j^2}.
\]

In the above theorem, \(\alpha_{ij}[1]\) and \(\alpha_{ij}[2]\) mean the first and second elements of the vector \(\alpha_{ij}\) in (2.9). If we let \(\nu \to \infty\), ie, \(X\) is the multivariate skew-normal, we have that \(a_{ij}^{1/\nu} \to 1\) and \(a_{ij}/\nu \to 1\). Since \(\rho_i \in (0, 1), i = 1, 2, \ldots, n,\) it follows from Theorem 2.4 that \(d_U(i, j) \to 0\) as \(\nu \to \infty\). In other words, if \(X\) follows the multivariate skew-normal distribution, any pair of \(X_i\) and \(X_j\) is asymptotically independent.

Suppose that \(X\) follows the multivariate Student \(t\) distribution, ie, \(\alpha = 0\). Then, (2.9) gives \(\alpha_{ij}[1] = \alpha_{ij}[2] = 0\), which implies \(\tau_{ij} = \beta_{ij} = 0\) for \(1 \leq i, j \leq n, i \neq j\). We then have \(a_{ij} = 1\), and the upper tail dependence of \(X_i\) and \(X_j\) is computed as follows:

\[
d_U(i, j) = 2 \left[ 1 - T\left(\frac{(1 - \rho_i \rho_j) \sqrt{\frac{v + 1}{\sqrt{1 - \rho_i^2 \rho_j^2}}}; v\right)\right].
\]

### 3 CONDITIONAL MONTE CARLO AND IMPORTANCE-SAMPLING SIMULATIONS

In many practical situations, it is essential to obtain \(\theta_n^* = \Pr\{L_n > \gamma\}\) for a large value of \(\gamma\), ie, the probability of a large loss occurring. However, \(\theta_n^*\) does not have closed form, and thus it is difficult to compute \(\theta_n^*\) based on \(((c_i, \rho_i, \bar{\rho}_i)_{i=1}^n, \alpha, \sigma, \nu)\). Instead of carrying out an exact computation, we are able to use the Monte Carlo algorithm for estimating \(\theta_n^*\). Let \((\eta(t), \xi(t), Z(t), \lambda(t))_{t=1}^N\) be independent random copies of \((\eta, \xi, Z, \lambda)\), and let \(L_n(t), t = 1, 2, \ldots, N,\) be the \(t\)th realized value of (2.1) through random sampling. The plain Monte Carlo estimator of \(\theta_n^*\) with \(N\) random samples is given by

\[
\bar{\theta}_{n,N} = \frac{1}{N} \sum_{t=1}^N \mathbb{1}\{L_n(t) > \gamma\}.
\]

Then, \(\bar{\theta}_{n,N}\) is an unbiased estimator of \(\theta_n^*\), and the mean squared error of \(\bar{\theta}_{n,N}\) is

\[
\text{Var}[\bar{\theta}_{n,N}] = \frac{1}{N} \text{Var}[\mathbb{1}(L_n > \gamma)],
\]

which converges to 0 as \(N \to 0\). With large values of \(N, \bar{\theta}_{n,N}\) gives a reliable estimate of \(\theta_n^*\).
As an alternative estimator of \( \hat{\theta}_{n,N} \), we can consider
\[
\hat{\theta}_{n,N} = \frac{1}{N} \sum_{i=1}^{N} E[\Pi(L_n^{(i)} > \gamma) \mid \mathcal{G}].
\] (3.3)
where \( \mathcal{G} \) is a suitable sub-\( \sigma \)-field of \( (\eta, \xi, Z, \lambda) \). Clearly, \( \hat{\theta}_{n,N} \) is also an unbiased estimator of \( \theta_n^* \), and its mean squared error is given by
\[
\text{Var}[\hat{\theta}_{n,N}] = \frac{1}{N} \text{Var}[E[\Pi(L_n > \gamma) \mid \mathcal{G}]].
\] (3.4)

Due to the variance reduction property of the conditional expectation, for any choice of \( \mathcal{G} \), we have that \( \text{Var}[\hat{\theta}_{n,N}] \leq \text{Var}[\hat{\theta}_{n,N}] \), i.e., the conditional Monte Carlo estimator is more efficient than the plain Monte Carlo estimator in terms of the mean squared error. However, for a conditional Monte Carlo estimator to be practical, it is essential that the value of \( E[\Pi(L_n > \gamma) \mid \mathcal{G}] \) is easily computable.

In the case of zero recovery rates, i.e., \( R_1 = \cdots = R_n = 0 \) with probability 1, Chan and Kroese (2010) proposed an efficient conditional Monte Carlo estimator by choosing the \( \sigma \)-field generated by \( (\eta, Z) \) as the sub-\( \sigma \)-field, \( \mathcal{G} \), in (3.3). They obtained a simple and easily computable form of \( E_{\lambda}[\Pi(L_n > \gamma) \mid \eta, Z] \), which is equal to \( \text{Pr}\{L_n > \gamma \mid \eta, Z\} \). We find that their method for obtaining the conditional expectation can be applied to our model by choosing the \( \sigma \)-field generated by \( (\eta, \xi, Z) \) as the sub-\( \sigma \)-field, \( \mathcal{G} \), in (3.3). In what follows, we let \( Y \) denote \( (\eta, \xi, Z) \). We obtain a simple form of \( E_{\lambda}[\Pi(L_n > \gamma) \mid Y] \) that is easily computable, as shown by the following theorem.

**Theorem 3.1** Let \( F_{\lambda}(x) \) be the cdf of \( \lambda \), and let \( W_{(1)} \leq W_{(2)} \leq \cdots \leq W_{(n)} \) be the order statistics of \( \{W_1, \ldots, W_n\} \), where
\[
W_i = \frac{\rho_i Z + \sqrt{1 - \rho_i^2} \eta_i}{x_i}, \quad i = 1, 2, \ldots, n.
\]
Suppose that \( Y = (\eta, \xi, Z) \), and \( \gamma > 0 \) is given. If
\[
\sum_{i=1}^{n} c_i (1 - R_i) \leq \gamma,
\]
then we let \( r = 0 \). Otherwise, we let \( r = \max\{W_{(k)}, 0\} \), where
\[
k = \max\left\{ 1 \leq l \leq n : \sum_{i=l}^{n} c_i (1 - R_{(i)}) > \gamma \right\}.
\]
Then, we have that
\[
E_{\lambda}[\Pi(L_n > \gamma) \mid Y] = F_{\lambda}(r^2 \mid Y).
\]
Proof. For the case that $\sum_{i=1}^{n} c_i (1 - R_i) \leq \gamma$, we can see that

$$\Pr\{L_n > \gamma \mid Y\} = 0,$$

since $L_n \leq \sum_{i=1}^{n} c_i (1 - R_i)$ for all default events of the obligors. Thus, the statement of the theorem is satisfied with $r = 0$.

For the other case, there exists $k$ such that

$$k = \max \left\{ 1 \leq l \leq n : \sum_{i=l}^{n} c(i)(1 - R(i)) > \gamma \right\}.$$

We denote by $(i)$, $i = 1, 2, \ldots, n$, the obligor corresponding to $W(i)$. Then, $c(i)(1 - R(i))$ is the monetary loss corresponding to the obligor $(i)$. The obligor $(i)$ defaults when $X(i) > x(i)$, or $\sqrt{\lambda} < W(i)$. Since $\{W(1), W(2), \ldots, W(n)\}$ are in their ascending order, the default of the obligor $(i)$ implies the defaults of the obligors $(i+1), (i+2), \ldots, (n)$, which means $L_n \geq \sum_{j=1}^{n} c(j)(1 - R(j))$, and vice versa. Then, from the definition of $k$, we can see that the event $\{L > \gamma\}$ occurs if and only if the obligor $(k)$ defaults. The latter event occurs when $\sqrt{\lambda} < W(k)$, which gives

$$\Pr\{L_n > \gamma \mid Y\} = F_\lambda(\max\{W(k), 0\}^2 \mid Y),$$

and the proof is complete.

Let

$$S(Y) = F(r^2 \mid Y).$$

Then, it follows from Theorem 3.1 that $\Pr\{L_n > \gamma\}$ can be estimated by the following conditional Monte Carlo estimator:

$$\hat{\theta}_{n,N} = \frac{1}{N} \sum_{i=1}^{N} S(Y^{(i)}), \quad (3.5)$$

where $Y^{(1)}, Y^{(2)}, \ldots, Y^{(N)}$ are independent and identically distributed copies of $Y$. The pdf of $Y$ is given by

$$f(y; \mu) = f_{SN}(z; 0, 1, \alpha) \prod_{i=1}^{n} f_{N}(\eta_i; 0, 1) f_{N}(\xi_i; 0, \sigma^2),$$

where $f_{SN}(\cdot; \mu, \sigma^2, \alpha)$ is the pdf of the skew-normal random variable with location parameter $\mu$, scale parameter $\sigma$ and shape parameter $\alpha$, and $f_{N}(\cdot; \mu, \sigma^2)$ is the pdf of normal random variable with mean $\mu$ and variance $\sigma^2$. Algorithm 3.2 summarizes the procedure described above.
Algorithm 3.2 (Conditional Monte Carlo)

(1) for $t = 1$ to $N$ do

(2) Obtain a sample $Y^{(t)} = (Z^{(t)}, \eta^{(t)}, \xi^{(t)})$

(3) for $i = 1$ to $n$ do

(4) compute $W_i, R_i$ from $Y^{(t)}$

(5) end for

(6) obtain the order $((1), \ldots, (n))$ so that $W_{(1)}, \ldots, W_{(n)}$ are in ascending order by sorting $W_1, \ldots, W_n$

(7) if $\sum_{i=1}^n c(i)(1 - R(i)) \leq \gamma$ then

(8) $\gamma \leftarrow 0$

(9) else

(10) $k \leftarrow \max\{l : \sum_{i=l}^n c(i)(1 - R(i)) > \gamma\}$

(11) $r \leftarrow \max\{W_k, 0\}$

(12) end if

(13) $S(Y^{(t)}) \leftarrow F(r^2)$

(14) end for

(15) return $\frac{1}{N} \sum_{t=1}^N S(Y^{(t)})$.

For fixed $x_1, x_2, \ldots, x_n$, if $\Pr\{L_n > \gamma \mid Y\} \notin \{0, 1\}$ with probability 1, then we have that

$$E_Y[\Var[\mathbb{1}(L_n > \gamma) \mid Y]] = E_Y[\Pr\{L_n > \gamma \mid Y\}(1 - \Pr\{L_n > \gamma \mid Y\})] > 0,$$

and, hence, we obtain the following theorem due to the law of total variance.

Theorem 3.3 The conditional Monte Carlo estimator $\hat{\theta}_{n,N}$ in (3.5) has strictly less squared error than the plain Monte Carlo estimator $\bar{\theta}_{n,N}$ if $\Pr\{L_n > \gamma \mid Y\} \notin \{0, 1\}$ with probability 1, i.e., $\Var[\hat{\theta}_{n,N}] < \Var[\bar{\theta}_{n,N}]$ for each $n$ and $N$. 

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It is possible to consider the \( \sigma \)-field generated by \((\eta, Z)\) as \( \mathcal{F} \) in the conditional Monte Carlo algorithm. In this case, we should compute the value of \( E_{\mathcal{F}, \lambda}[\mathbb{I}(L_n > \gamma) \mid \eta, Z] \). Using \( E[c_{(i)}(1 - R_{(i)})] \) instead of \( c_{(i)}(1 - R_{(i)}) \), we can construct another conditional Monte Carlo algorithm. However, we can confirm empirically that the computational complexity in obtaining \( E[c_{(i)}(1 - R_{(i)})] \) diminishes the gain from using the conditional Monte Carlo algorithm.

Although the conditional Monte Carlo estimator performs better than the plain Monte Carlo estimator, its performance can be improved by applying the importance sampling technique, i.e., by sampling \( Y \) not from \( f(y; u) \) but from a different distribution. More specifically, we consider the following importance sampling estimator, called the conditional Monte Carlo with cross entropy:

\[
\frac{1}{N} \sum_{t=1}^{N} S(Y^{(t)}) \frac{f(Y^{(t)}; u)}{g(Y^{(t)})},
\]

where \( Y^{(1)}, \ldots, Y^{(N)} \) are sampled independently from \( g(y) \), an appropriately chosen density for sampling.

**Algorithm 3.4** (Conditional Monte Carlo with cross entropy)

1. for \( t = 1 \) to \( M \) do
2. generate \( Y^{(t)} = (Z^{(t)}, \eta^{(t)}, \xi^{(t)}) \) from \( f(\cdot; u) \)
3. end for
4. compute \( v^* = (\mu_{z}^*, V_{z}^*, \mu_{\eta}^*, V_{\eta}^*, \mu_{\xi}^*, V_{\xi}^*, \alpha^*) \) such that
   \[
   \sum_{t=1}^{M} S(Y^{(t)}) \log f(Y^{(t)}; v)
   \]
   is maximized at \( v = v^* \)
5. for \( t = 1 \) to \( N \) do
6. generate \( Y^{(t)} \) from \( f(\cdot; v^*) \)
7. compute \( S(Y^{(t)}) \) using the procedure in steps 3 to 13 of Algorithm 3.2
8. end for
9. return \( \frac{1}{N} \sum_{t=1}^{N} S(Y^{(t)}) \frac{f(Y^{(t)}; u)}{f(Y^{(t)}; v^*)} \)
It is ideal to use $g(y)^*$ as the zero-variance proposal density given by

$$g(y)^* \propto S(y) f(y; \mu).$$

However, the exact form of the density function is unknown. Following Chan and Kroese (2010), we consider the possible parametric family $\mathcal{F}$ of proposal densities, which contains the original pdf $f(y; \mu)$, and adopt the cross-entropy method for choosing the pdf that minimizes Kullback–Leibler divergence with the zero-variance proposal density. Here, we let

$$\mathcal{F} = \left\{ f(y; \nu) = f_{SN}(z; \mu_z, V_z, \alpha_z) \prod_{i=1}^{n} f_{N}(\eta_i; \mu_{\eta}, V_{\eta}) f_{N}(\xi_i; \mu_{\xi}, V_{\xi}) \right\}. \quad (3.7)$$

In practice, we find the parameter $\nu^*$ such that

$$\nu^* = \arg \max_{\nu} \sum_{t=1}^{M} S(Y^{(t)}) \log f(Y^{(t)}; \nu),$$

where $Y^{(t)}$, $t = 1, 2, \ldots, M$, is a random sample from $f(y; \mu)$, and use $f(y; \nu^*)$ instead of $g(y)^*$.

Part of the solution in $\nu^*$ can be computed analytically by using

$$\mu^*_\eta = \frac{\sum_{t=1}^{M} S(Y^{(t)}) \sum_{i=1}^{n} \eta_i^{(t)}}{n \sum_{t=1}^{M} S(Y^{(t)})},$$

$$V^*_\eta = \frac{\sum_{t=1}^{M} S(Y^{(t)}) \sum_{i=1}^{n} (\eta_i^{(t)} - \mu^*_\eta)^2}{n \sum_{t=1}^{M} S(Y^{(t)})},$$

$$\mu^*_\xi = \frac{\sum_{t=1}^{M} S(Y^{(t)}) \sum_{i=1}^{n} \xi_i^{(t)}}{n \sum_{t=1}^{M} S(Y^{(t)})},$$

$$V^*_\xi = \frac{\sum_{t=1}^{M} S(Y^{(t)}) \sum_{i=1}^{n} (\xi_i^{(t)} - \mu^*_\xi)^2}{n \sum_{t=1}^{M} S(Y^{(t)})}.$$ 

The remainder of the solution in $\nu^*$ is given by

$$(\mu^*_z, V^*_z, \alpha^*_z) = \arg \max_{\mu, V > 0, \alpha} \sum_{t=1}^{M} S(Y^{(t)}) \log f_{SN}(z^{(t)}; \mu, V, \alpha),$$

and we use a quasi-Newton method with a box constraint (Byrd et al 1995) to obtain the solution.

Once we have obtained the optimal proposal density $f(y; \nu^*)$, the conditional Monte Carlo with the cross-entropy estimator in (3.6) with $g(Y^{(t)}) = f(Y^{(t)}; \nu^*)$ gives an estimated value of $Pr\{L > \gamma\}$. The procedure is summarized in Algorithm 3.4.
4 ASYMPTOTIC ANALYSIS

Due to Theorem 3.3, the conditional Monte Carlo estimator $\hat{\theta}_{n,N}$ is a more efficient estimator of $\theta^*_n$ than the plain Monte Carlo estimator $\tilde{\theta}_{n,N}$ for each $n$ and $N$ in terms of mean squared error. In this section, we compare the efficiency of the two estimators asymptotically when the event $\{L_n > \gamma\}$ is rare. More specifically, we consider the situation when $\Pr\{L_n > \gamma\}$ goes to zero with diverging $x_1, x_2, \ldots, x_n$ and $\gamma$ as $n \to \infty$. We let $x_i = a_i h(n)$, $i = 1, 2, \ldots, n$, in (2.1) and $\gamma = nb$, where $a_i$ and $b$ are positive constants and $h(n)$ is an infinitely increasing function. Then, the total loss in (2.1) can be rewritten as

$$L_n = \sum_{i=1}^{n} c_i (1 - R_i) \mathbb{I}\{X_i > a_i h(n)\}.$$

For the case of zero recovery rates and a subexponentially increasing $h(n)$, Bassamboo et al (2008) showed that $\Pr\{L_n > nb\} \to 0$ as $n \to \infty$, and that the convergence rate of $\Pr\{L_n > nb\}$ is proportional to $h(n)^{-2\nu}$, i.e., for a constant $K$,

$$\lim_{n \to \infty} h(n)^{2\nu} \Pr\{L_n > nb\} = K \quad \text{as} \quad n \to \infty. \quad (4.1)$$

In this section, we extend the above result to our proposed model, i.e., the case of dependent random recovery rates, and show that the conditional Monte Carlo estimator $\hat{\theta}_{n,N}$ is relatively efficient, i.e., for $N \geq 1$,

$$\lim_{n \to \infty} \left(1 - \frac{\Var[\hat{\theta}_{n,N}]}{\Var[\tilde{\theta}_{n,N}]}\right) = 1.$$

In what follows, we assume that $h(n)$ is a subexponentially increasing function. We impose the following assumptions on $\{(a_i, c_i, \mu_i, \rho_i, \tilde{\rho}_i)\}_{i=1}^{n}$, as in Bassamboo et al (2008).

**Assumption 4.1** The nonnegative sequence $\{(a_i, c_i, \mu_i, \rho_i, \tilde{\rho}_i)\}_{i=1}^{n}$ takes values in a finite set $\mathcal{V}$, with cardinality $|\mathcal{V}|$. An obligor described by $(a_j, c_j, \mu_j, \rho_j, \tilde{\rho}_j)$ is called a type-$j$ obligor ($j = 1, 2, \ldots, |\mathcal{V}|$). The proportion of type-$j$ obligors in the portfolio converges to $q_j > 0$ as $n \to \infty$, so that $\sum_{j \in \mathcal{V}} q_j = 1$.

Under Assumption 4.1, we obtain the following lemma for the large deviation bound of the average loss, $L_n/n$.

**Lemma 4.2** Let

$$r(w, z) = \lim_{n \to \infty} \sum_{i=1}^{n} r_i^{(n)}(w, z)/n,$$
where

\[ r_i^{(n)}(w, z) = E[c_i(1 - R_i) \mathbb{1}\{X_i > a_i h(n)\} | \sqrt{\lambda} = w / h(n), Z = z] \]

for \( w > 0, z \in \mathbb{R} \). Under Assumption 4.1, there exists \( N \in \mathbb{N} \) such that

\[
\Pr \left\{ \left| \frac{L_n}{n} - r(w, z) \right| > \varepsilon \left| \sqrt{\lambda} = \frac{w}{h(n)}, Z = z \right\} \leq 2 \exp(-Cn)
\]

for all \( n > N \), where \( C \) is a constant independent of \( w \) and \( z \).

**Proof** Without loss of generality, we assume that \( |V| = 1 \), ie, all of the obligors are of the same type. For the case of \( |V| > 1 \), the proof can be done by modifying the followings. Given \( Z = z, R_i \) and \( X_i \) are conditionally independent. Then, we have

\[ r_i^{(n)}(w, z) = E[c_i(1 - R_i) | Z = z]E[\mathbb{1}\{X_i > a_i h(n)\} | \sqrt{\lambda} = w / h(n), Z = z]. \]

Note that the event \( \{X_i > a_i h(n) | \sqrt{\lambda} = w / h(n), Z = z\} \) is equivalent to the event \( \{\rho_i z + \sqrt{1 - \rho_i^2} \eta_i > a_i w\} \), which implies that the former does not depend on \( n \). Clearly, \( E[c_i(1 - R_i)|Z = z] \) also does not depend on \( n \). Since all of the obligors are of the same type, we can see that \( r_1^{(n)}(w, z) = \cdots = r_n^{(n)}(w, z) \). Thus, we have that for \( i = 1, 2, \ldots, n \),

\[ r_i^{(n)}(w, z) = r(w, z). \]

Then, it follows that

\[
\Pr \left\{ \left| \frac{L_n}{n} - r(w, z) \right| > \varepsilon \left| \sqrt{\lambda} = \frac{w}{h(n)}, Z = z \right\} = \Pr \left\{ \left| \frac{L_n}{n} - \frac{1}{n} \sum_{i=1}^{n} r_i^{(n)}(w, z) \right| > \varepsilon \left| \sqrt{\lambda} = \frac{w}{h(n)}, Z = z \right\}. \]

The random variable \( c_i(1 - R_i) \mathbb{1}\{X_i > a_i h(n)\} \) is bounded with a value in \([0, c_i], i = 1, 2, \ldots, n\). Then, Hoeffding’s inequality says that

\[
\Pr \left\{ \left| \frac{L_n}{n} - r(w, z) \right| > \varepsilon \left| \sqrt{\lambda} = \frac{w}{h(n)}, Z = z \right\} \leq 2 \exp \left\{ -\frac{2n^2 \varepsilon^2}{\sum_{i=1}^{n} c_i^2} \right\}. \]

Since \( c_1 = c_2 = \cdots = c_n \), we have that

\[
\Pr \left\{ \left| \frac{L_n}{n} - r(w, z) \right| > \varepsilon \left| \sqrt{\lambda} = \frac{w}{h(n)}, Z = z \right\} \leq 2 \exp \left\{ -\frac{2n^2 \varepsilon^2}{c_1^2} \right\},
\]

which completes the proof. \( \square \)
Lemma 4.2 says that $L_n/n$ converges to $r(w, z)$ uniformly with an exponentially decreasing rate. It can be easily checked that $r(w, z)$ has the same monotone properties as in Bassamboo et al (2008) even when $R_1, R_2, \ldots, R_n$ are dependent random variables. The uniformly exponential bound of $|L_n/n - r(w, z)|$ and the monotone property of $r(w, z)$ play a key role in showing the convergence rate. As a consequence of these two properties, we can obtain the following lemma in a similar manner to Bassamboo et al (2008). We omit the proof here.

**Lemma 4.3** Under Assumption 4.1, there exists $K' > 0$ such that, for $0 < b < \sum_{j \leq |\mathcal{V}|} q_j c_j$,

$$\lim_{n \to \infty} h(n)2^n \Pr\{L_n > nb\} = K' > 0 \quad \text{as } n \to \infty.$$ 

Now, it can be shown that the conditional Monte Carlo estimator $\hat{\theta}_{n,N}$ is relatively efficient.

**Theorem 4.4** Under Assumption 4.1, for $0 < b < \sum_{j \leq |\mathcal{V}|} q_j c_j$ and $N \geq 1$,

$$\lim_{n \to \infty} \left(1 - \frac{\text{Var}[\hat{\theta}_{n,N}]}{\text{Var}[\hat{\theta}_{n,N}]}\right) = 1.$$ 

**Proof** We denote the relative efficiency of our proposed algorithm by $F(n)$, ie,

$$F(n) = 1 - \frac{\text{Var}[\hat{\theta}_{n,N}]}{\text{Var}[\tilde{\theta}_{n,N}]}.$$ 

(4.2)

It follows from (3.5) that

$$\text{Var}[\hat{\theta}_{n,N}] = \frac{\text{Var}[S(Y)]}{N}.$$ 

Equation (3.2) says that $\text{Var}[\tau_{n,N}] = \text{Var}[I(L_n > nb)]/N$. Since $\Pr\{I(L_n > nb)\} = E[S(Y)]$, we have

$$\text{Var}[\tilde{\theta}_{n,N}] = \frac{E[S(Y)](1 - E[S(Y)])}{N}.$$ 

Substituting the above two equations into (4.2), we have

$$F(n) = \frac{E[S(Y)] - E[S^2(Y)]}{E[S(Y)] - E[S(Y)]^2},$$

or, equivalently,

$$F(n) = \frac{1 - E[S^2(Y)]/E[S(Y)]}{1 - E[S(Y)]}.$$
Since $E[S(Y)] \to 0$ as $n \to \infty$ by Lemma 4.3, in order to show that $\lim_{n \to \infty} F(n) = 1$, it suffices to show that
\[
\limsup_{n \to \infty} \frac{E[S^2(Y)]}{E[S(Y)]^2} < \infty.
\]

Let
\[
L_{\text{max}} = \sum_{i=1}^{n} c_i \mathbb{I}\{X_i > x_i\} \quad \text{and} \quad S_{\text{max}}(Y) = \Pr\{L_{\text{max}} > nb \mid Y\},
\]

ie, the total loss and the probability of a large loss occurring under zero recovery rates, respectively. Clearly, $S_{\text{max}}(Y) \geq S(Y)$ almost surely, which means $E[S_{\text{max}}^2(Y)] \geq E[S^2(Y)]$. Then, we have
\[
\limsup_{n \to \infty} \frac{E[S^2(Y)]}{E[S(Y)]^2} \leq \limsup_{n \to \infty} \frac{E[S_{\text{max}}^2(Y)]}{E[S_{\text{max}}(Y)]^2} \limsup_{n \to \infty} \frac{E[S_{\text{max}}(Y)]^2}{E[S(Y)]^2},
\]

The term $E[S_{\text{max}}^2(Y)]/E[S_{\text{max}}(Y)]^2$ on the right-hand side of the above equation is an efficiency measure of the conditional Monte Carlo with zero recovery, which is considered by Chan and Kroese (2010). They showed that
\[
\limsup_{n \to \infty} \frac{E[S_{\text{max}}^2(Y)]}{E[S_{\text{max}}(Y)]^2} < \infty.
\]

Now it remains to show that $\limsup_{n \to \infty} E[S_{\text{max}}(Y)]^2/E[S(Y)]^2 < \infty$, or, equivalently,
\[
\limsup_{n \to \infty} \frac{\Pr\{L_{\text{max}} > nb\}}{\Pr\{L > nb\}} < \infty.
\]

Equation (4.1) and Lemma 4.3 yield
\[
\lim_{n \to \infty} \frac{\Pr\{L_{\text{max}} > nb\}}{\Pr\{L > nb\}} = \frac{K}{K'},
\]

which completes the proof.

5 NUMERICAL STUDY

5.1 Simulation results

In this subsection, we demonstrate the performance of the conditional Monte Carlo (condMC) and conditional Monte Carlo with cross entropy (condMC-CE) in estimating large portfolio loss probabilities when the credit portfolio has random recovery. For comparison, we use parameter values similar to those in Chan and Kroese (2010).
TABLE 1 Performance of condMC and condMC-CE for various degrees of freedom.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>CMC</th>
<th>CE</th>
<th>CMC</th>
<th>CE</th>
<th>CMC</th>
<th>CE</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2.673E–2</td>
<td>2.671E–2</td>
<td>2.683E+1</td>
<td>1.581E+2</td>
<td>0.118</td>
<td>0.049</td>
</tr>
<tr>
<td>8</td>
<td>3.961E–3</td>
<td>3.970E–3</td>
<td>4.043E+1</td>
<td>4.007E+2</td>
<td>0.249</td>
<td>0.079</td>
</tr>
<tr>
<td>12</td>
<td>9.728E–4</td>
<td>9.740E–4</td>
<td>5.038E+1</td>
<td>9.274E+2</td>
<td>0.451</td>
<td>0.105</td>
</tr>
<tr>
<td>16</td>
<td>3.204E–4</td>
<td>3.219E–4</td>
<td>5.781E+1</td>
<td>1.762E+3</td>
<td>0.733</td>
<td>0.133</td>
</tr>
<tr>
<td>20</td>
<td>1.299E–4</td>
<td>1.304E–4</td>
<td>5.760E+1</td>
<td>3.118E+3</td>
<td>1.153</td>
<td>0.157</td>
</tr>
</tbody>
</table>

CMC denotes conditional Monte Carlo. CE denotes conditional Monte Carlo with cross entropy.

All simulations are based on one million samples ($N = 1000000$ for both estimators, condMC and condMC-CE, defined in (3.5) and (3.6), respectively). Fifty thousand samples are generated to find the optimal parameters for condMC-CE, ie, $M = 50000$. For ease of analysis, we assume that $\rho_1, \rho_2, \ldots, \rho_n$ have the same value as $\rho$, and $\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_n$ have the same value as $\tilde{\rho}$ in all simulations. We also assume that $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.25$ for the same reason. We fix the nonvarying parameters with the following values:

\[
\begin{align*}
 n &= 250, & \sigma_\xi &= 1, & \rho &= 0.1, & \tilde{\rho} &= 2, & \alpha &= 5, \\
 h(n) &= n^{1/4}, & b &= 0.40, & \mu &= -2, & \nu &= 12.
\end{align*}
\]

To evaluate the performance of the proposed estimators, we consider three measures. First, variance reduction relative to plainMC is considered as a major measure of efficiency. To obtain the variance of the plainMC estimator, the probability of a large loss occurring, $p = \Pr\{L > nb\}$, is approximated by the estimate $\hat{p}$, the condMC-CE estimate is given by (3.6) and the variance of the plainMC estimator is calculated as $\hat{p}(1 - \hat{p})/N$. We also consider the relative error of the estimators, defined as the ratio of the standard deviation of the estimator to $p$. However, we do not know either the exact value of $p$ or that of the standard deviation. Thus, the relative error is approximated as $S/(\hat{p}\sqrt{N})$, where $S$ is the sample variance of simulated samples $\{S(Y(t)) : t = 1, \ldots, N\}$ for condMC, and $\{S(Y(t)f(Y(t); u)/f(Y(t); u^*) : t = 1, \ldots, N\}$ for condMC-CE, respectively. The last measure is the half-length of the 95% confidence interval (95% CI) for $p$, $\hat{p} - (1.96)S/\sqrt{N}, \hat{p} + (1.96)S/\sqrt{N}$.

Table 1 shows the performance of condMC and condMC-CE with various values of the degrees of freedom, $\nu$, in the Gamma distributions for $\lambda$. The results of our simulation indicate that the variance reduction of the two methods increases as the degrees of freedom increase, and that the half-length of the 95% CI decreases as...
TABLE 2   Performance of condMC and condMC-CE for various values of the skewness parameter.

<table>
<thead>
<tr>
<th>Probability estimate</th>
<th>Variance reduction</th>
<th>Relative error (%)</th>
<th>Half-length of 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>CMC</td>
<td>CE</td>
<td>CMC</td>
</tr>
<tr>
<td>0</td>
<td>4.838E-4</td>
<td>4.880E-4</td>
<td>5.063E+1</td>
</tr>
<tr>
<td>3</td>
<td>9.686E-4</td>
<td>9.702E-4</td>
<td>5.044E+1</td>
</tr>
<tr>
<td>7</td>
<td>9.738E-4</td>
<td>9.748E-4</td>
<td>5.036E+1</td>
</tr>
</tbody>
</table>

CMC denotes conditional Monte Carlo. CE denotes conditional Monte Carlo with cross entropy.

TABLE 3   Performance of condMC and condMC-CE for various values of $\rho$.

<table>
<thead>
<tr>
<th>Probability estimate</th>
<th>Variance reduction</th>
<th>Relative error (%)</th>
<th>Half-length of 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>CMC</td>
<td>CE</td>
<td>CMC</td>
</tr>
<tr>
<td>0.1</td>
<td>3.274E-4</td>
<td>3.283E-4</td>
<td>6.384E+1</td>
</tr>
<tr>
<td>0.2</td>
<td>6.828E-3</td>
<td>6.819E-3</td>
<td>6.805E+0</td>
</tr>
<tr>
<td>0.3</td>
<td>3.501E-2</td>
<td>3.499E-2</td>
<td>2.999E+0</td>
</tr>
<tr>
<td>0.4</td>
<td>8.674E-2</td>
<td>8.673E-2</td>
<td>2.217E+0</td>
</tr>
</tbody>
</table>

CMC denotes conditional Monte Carlo. CE denotes conditional Monte Carlo with cross entropy.

the degrees of freedom increase. The relative errors remain in a reasonable range. From these observations, we can see that performance of the two estimators improves greatly as the degrees of freedom increase. The variance reduction of condMC-CE is much larger than that of condMC, and the relative error is more stable, as it is less than 1.0% for all settings. A similar pattern of efficiency gain is observed when comparing confidence intervals. Half-lengths of intervals obtained by condMC are approximately ten times wider than those obtained by condMC-CE.

Table 2 shows the performance of condMC and condMC-CE with various values of $\alpha$. Recall that the distribution of $Z$ with positive $\alpha$ has a right-skewed tail. Our simulation result shows that both condMC and condMC-CE provide an estimator with smaller variance than the plainMC, and that the gain in the efficiency of condMC-CE is still much larger than condMC for all three measures. Table 2 indicates that the reduction does not greatly depend on the value of $\alpha$. Since $\alpha$ does not affect the tail property of $X_i$, the simulation result is supported by Theorem 4.4, in which the relative efficiency does not require the condition on $\alpha$.

Table 3 shows the performance of condMC and condMC-CE with various values of $\rho$. Our simulation result shows that the variance reduction of the two estimators
TABLE 4  Performance of condMC and condMC-CE for various values of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>CMC Probability estimate</th>
<th>Variance reduction</th>
<th>Relative error (%)</th>
<th>Half-length of 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CMC</td>
<td>CE</td>
<td>CMC</td>
<td>CE</td>
</tr>
<tr>
<td>100</td>
<td>1.356E−2</td>
<td>1.351E−2</td>
<td>6.794E+0</td>
<td>4.434E+1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.330</td>
<td>0.129</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8.739E−5</td>
<td>3.421E−5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.451</td>
<td>0.105</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8.614E−6</td>
<td>2.008E−6</td>
</tr>
<tr>
<td>500</td>
<td>1.189E−4</td>
<td>1.180E−4</td>
<td>3.143E+2</td>
<td>1.291E+4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.519</td>
<td>0.081</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.201E−6</td>
<td>1.874E−7</td>
</tr>
</tbody>
</table>

CMC denotes conditional Monte Carlo. CE denotes conditional Monte Carlo with cross entropy.

TABLE 5  Performance of condMC and condMC-CE for various values of $b$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>CMC Probability estimate</th>
<th>Variance reduction</th>
<th>Relative error (%)</th>
<th>Half-length of 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CMC</td>
<td>CE</td>
<td>CMC</td>
<td>CE</td>
</tr>
<tr>
<td>0.35</td>
<td>7.975E−3</td>
<td>7.988E−3</td>
<td>2.128E+1</td>
<td>2.098E+2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.243</td>
<td>0.077</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3.797E−5</td>
<td>1.209E−5</td>
</tr>
<tr>
<td>0.40</td>
<td>9.728E−4</td>
<td>9.740E−4</td>
<td>5.043E+1</td>
<td>9.282E+2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.451</td>
<td>0.105</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8.614E−6</td>
<td>2.008E−6</td>
</tr>
<tr>
<td>0.45</td>
<td>9.006E−5</td>
<td>9.020E−5</td>
<td>1.242E+2</td>
<td>5.584E+3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.945</td>
<td>0.141</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.670E−6</td>
<td>2.491E−7</td>
</tr>
</tbody>
</table>

CMC denotes conditional Monte Carlo. CE denotes conditional Monte Carlo with cross entropy.

Increases as the value of $\rho$ decreases, and that the half-length of the 95% CI increases as $\rho$ decreases. The relative errors remain in a reasonable range, as shown in the table. In contrast to the result in Table 1, the performance of the two estimators shows a large gain as $\rho$ decreases. However, the gain in efficiency of condMC-CE is still much larger than that of condMC for all three measures.

Table 4 shows the performance of the proposed estimators for different numbers of obligors. Our simulation shows that the gain in performance of the two estimators increases with the number of obligors. The variance reduction of condMC-CE is much larger than that of condMC, and the relative error of condMC-CE is much more stable. Table 5 shows a similar pattern in the change in efficiency of the two estimators when considering different values for $b$ in $Pr\{L > bn\}$. A higher value of $b$ gives a greater reduction in variance and a narrower confidence interval.

Table 6 illustrates performance with various values of $\mu$. As the value of $\mu$ increases, the expected recovery rates increase, and the probability of the loss exceeding $nb$ reduces.

Throughout the five considered simulations, condMC and condMC-CE are more efficient than plainMC in estimating the probability of rare events. The smaller the estimated probabilities, the larger the variance reduction in condMC and condMC-CE.
TABLE 6  Performance of condMC and condMC-CE for various values of $\mu$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Probability estimate</th>
<th>Variance reduction</th>
<th>Relative error (%)</th>
<th>Half-length of 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CMC</td>
<td>CE</td>
<td>CMC</td>
<td>CE</td>
</tr>
<tr>
<td>−0.30</td>
<td>1.001E−3</td>
<td>1.002E−3</td>
<td>5.173E+1</td>
<td>8.740E+2</td>
</tr>
<tr>
<td>−0.25</td>
<td>9.922E−4</td>
<td>9.935E−4</td>
<td>5.130E+1</td>
<td>8.912E+2</td>
</tr>
<tr>
<td>−0.20</td>
<td>9.728E−4</td>
<td>9.740E−4</td>
<td>5.038E+1</td>
<td>9.274E+2</td>
</tr>
<tr>
<td>−0.15</td>
<td>9.389E−4</td>
<td>9.400E−4</td>
<td>4.884E+1</td>
<td>9.911E+2</td>
</tr>
<tr>
<td>−0.10</td>
<td>8.879E−4</td>
<td>8.886E−4</td>
<td>4.665E+1</td>
<td>1.089E+3</td>
</tr>
</tbody>
</table>

CMC denotes conditional Monte Carlo. CE denotes conditional Monte Carlo with cross entropy.

TABLE 7  Computation time of condMC and condMC-CE with 1 000 000 iterations.

<table>
<thead>
<tr>
<th></th>
<th>plainMC</th>
<th>condMC</th>
<th>condMC-CE</th>
<th>Estimation for cross entropy*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elapsed time (seconds)</td>
<td>170.53</td>
<td>289.76</td>
<td>406.19</td>
<td>17.51</td>
</tr>
<tr>
<td>Relative time</td>
<td>1.00</td>
<td>1.70</td>
<td>2.38</td>
<td>0.10</td>
</tr>
</tbody>
</table>

* Sample size for parameter estimation is 50 000.

This result supports Theorem 4.4, which implies that the relative efficiency goes to 1 as the true probability goes to 0. All the programs for the simulation study were written using R software (see www.r-project.org). Regarding computation time for simulations, Table 7 shows the time elapsed for each algorithm with one set of parameters. The elapsed time is based on 1 000 000 samples.

As shown in Table 7, the computation times for condMC and condMC-CE are 1.70 and 2.38 times longer than plainMC, respectively. The additional costs in computation time of the two conditional Monte Carlo methods are negligible considering the huge gain in efficiency. There is also a very minor computational cost for parameter estimation in the cross-entropy method. The computer used for our simulation study was running an Intel Xeon X5650 CPU at 2.67 GHz, with 48 GB memory and a Microsoft Windows 7 operating system.

5.2 Model calibration

The proposed model has several parameters characterizing the stochastic properties of the associated defaults and recovery rates. In this subsection, we discuss the calibration of the model parameters. For our calibration, we assume that the obligors are classified by their credit ratings and characteristics such as industrial classification. Let obligor $i$ be a class $i$ obligor for $i = 1, 2, \ldots, n$. Then, the default
probability \( p_i, i = 1, 2, \ldots, n \), is estimated to be the historical default probability of the class \( i \) obligors in each time horizon, and the joint default probability \( p_{ij}, 1 \leq i, j \leq n \), is estimated to be the historical joint default probability of the class \( i \) and class \( j \) obligors in each time horizon. A detailed discussion of the estimation method using historical data can be found in Nagpal and Bahar (2001). The value of \( \tau(R_i, R_j), 1 \leq i, j \leq n \), can be estimated in the same manner. The value of \( E[(1 - R_i)I(X_i > x_i)], i = 1, 2, \ldots, n \), is estimated from the credit default swap spread. Using these estimated values, and under an appropriate assumption of the value of \( v \), the model parameters can be calibrated by the least squares method.

Let \( \rho = (\rho_1, \ldots, \rho_n) \) and let \( x = (x_1, \ldots, x_n) \). The true model parameters \((\alpha^*, v^*, \rho^*, x)\) that explain the estimated values of default probability, \( p_i, i = 1, 2, \ldots, n \), and the joint default probability, \( p_{ij}, 1 \leq i, j \leq n \), can be calibrated using the least squares method. Note that \( \Pr(X_i > x_i), i = 1, 2, \ldots, n \), is a function of \( \alpha, v, \rho_i \) and \( x_i \), and is denoted by \( g_1(\alpha, v, \rho_i, x_i) \). Similarly, \( \Pr(X_i > x_i, X_j > x_j), 1 \leq i, j \leq n \), is denoted by \( g_2(\alpha, v, \rho_i, x_i, \rho_j, x_j) \). Then, we consider the loss function given by

\[
\mathcal{L}(\alpha, v, \rho, x) = \sum_{i=1}^{n} (p_i - g_1(\alpha, v, \rho_i, x_i))^2 + \sum_{1 \leq i < j \leq n} (p_{ij} - g_2(\alpha, v, \rho_i, x_i, \rho_j, x_j))^2.
\]

We select \((\alpha, v, \rho, x)\), minimizing \( \mathcal{L}(\alpha, v, \rho, x) \) as the values of the true model parameters.

We can also use the least squares method to calibrate the parameters \( \mu_i, \tilde{\rho}_i \) and \( \sigma_i \), \( i = 1, 2, \ldots, n \). When \( \sigma_i = 1 \) for all \( i \), we illustrate the procedure for calibrating \( \tilde{\rho}_i \) and \( \mu_i \) for \( i = 1, \ldots, n \). Let \( \tau_{ij}, 1 \leq i, j \leq n \), be the estimated value of \( \tau(R_i, R_j) \), and let

\[
g_3(\tilde{\rho}_i, \tilde{\rho}_j) = 4 \int_{\mathbb{R}^2} \Phi\left( \frac{\tilde{\rho}_i x}{\sqrt{2}} \right) \Phi\left( \frac{\tilde{\rho}_j x}{\sqrt{2}} \right) f_{SN}(y; \alpha^*) f_{SN}(x + y; \alpha^*) \, dy \, dx - 1,
\]

where \( \alpha^* \) is the true parameter of \( \alpha \). The value of \( \alpha^* \) is estimated by the above method. We select \( \tilde{\rho}^* = (\tilde{\rho}^*_1, \ldots, \tilde{\rho}^*_n) \) as the true model parameter, \( \tilde{\rho} \), to minimize the following loss function:

\[
\mathcal{L}(\tilde{\rho}) = \sum_{1 \leq i < j \leq n} (\tau_{ij} - g_3(\tilde{\rho}_i, \tilde{\rho}_j))^2. \tag{5.1}
\]

The minimizer of \( \mathcal{L}(\tilde{\rho}) \) is obtained by the Newton–Raphson method in which computations of the gradient vector and the Hessian matrix of \( \mathcal{L}(\tilde{\rho}) \) are required. These two terms can be computed since the first and second derivatives of \( g_3(\tilde{\rho}_i, \tilde{\rho}_j) \) are
FIGURE 3 Calibration error of $\tau_1, \tau_2, \ldots, \tau_n$ for each $\alpha$.

obtained via $\tau(R_i, R_j)$ in Theorem 2.2:

$$
\frac{\partial g_3(\tilde{\rho}_i, \tilde{\rho}_j)}{\partial \rho_i} = \int_{\mathbb{R}^2} \frac{x}{\sqrt{2}} \phi \left( \frac{\tilde{\rho}_i x}{\sqrt{2}} \right) \phi \left( \frac{\tilde{\rho}_j x}{\sqrt{2}} \right) f_{SN}(y; \alpha^*) f_{SN}(x + y; \alpha^*) \, dy \, dx,
$$

$$
\frac{\partial^2 g_3(\tilde{\rho}_i, \tilde{\rho}_j)}{\partial \rho_i \partial \rho_j} = \int_{\mathbb{R}^2} \frac{x^2}{2} \phi \left( \frac{\tilde{\rho}_i x}{\sqrt{2}} \right) \phi \left( \frac{\tilde{\rho}_j x}{\sqrt{2}} \right) f_{SN}(y; \alpha^*) f_{SN}(x + y; \alpha^*) \, dy \, dx,
$$

$$
\frac{\partial^2 g_3(\tilde{\rho}_i, \tilde{\rho}_j)}{\partial \rho_i^2} = -\int_{\mathbb{R}^2} \frac{\tilde{\rho}_i x^3}{2^{3/2}} \phi \left( \frac{\tilde{\rho}_i x}{\sqrt{2}} \right) \phi \left( \frac{\tilde{\rho}_j x}{\sqrt{2}} \right) f_{SN}(y; \alpha^*) f_{SN}(x + y; \alpha^*) \, dy \, dx.
$$

Note that $\mathcal{L}(\tilde{\rho})$ is not a convex function, and the initial values for applying the Newton–Raphson method should be set with care. In our experience, a good convergence is achieved with the initial value $(1, \ldots, 1)^n \in \mathbb{R}^n$. When other parameters, including $\tilde{\rho}$, are given, $E[(1 - R_i)I(X_i > x_i)]$ is a function of $\mu_i$. Since $E[(1 - R_i)I(X_i > x_i)]$ is strictly monotone, $\mu_i$ can be calibrated with the observed value of $E[(1 - R_i)I(X_i > x_i)]$.

To see how the calibration method described above can work well, we generated random values of the true model parameters $(\tilde{\rho}_1, \ldots, \tilde{\rho}_n)$, and computed the values of $\tau_{ij} = \tau(R_i, R_j)$, $1 \leq i, j \leq n$, using Theorem 2.2. With these values of $\tau_{ij}$ and the initial values $\tilde{\rho}_0 = (1, \ldots, 1)$, we obtained the calibrated value of $\tilde{\rho}^*$ by minimizing $\mathcal{L}(\tilde{\rho})$. We set $n = 5$, and the true values of $(\tilde{\rho}_1, \ldots, \tilde{\rho}_5)$ were generated on $[1, 5]^5$, ...
uniformly, for each simulation. The simulations were repeated 100 times for each $\alpha^* = 0, 3, 7$. Figure 3 shows the calibration errors in terms of the Euclidean distance between the true value $(\tilde{\rho}_1, \ldots, \tilde{\rho}_5)$ and the calibrated one. In Figure 3, we can see that the calibration for $\tilde{\rho}$ works well if $\alpha^*$ is well calibrated. The calibration error slightly increases as $\alpha^*$ decreases.

6 CONCLUDING REMARKS

We proposed a probability model for portfolio credit risk with random recovery. We adopted the skewed $t$-copula function for modeling the extremal dependence between defaults, and adopted the skewed normal copula function for dependent random recovery rates. By introducing a common risk factor into the two copula functions, the dependent structure of the defaults and that of the random recovery rates were incorporated. We extended the traditional $t$-copula model by incorporating dependent random recovery rates. Thus, our proposed model is more flexible, with a wider range of applications for portfolio credit risk.

We obtained Kendall’s tau correlations for any pair of obligor defaults and random recovery rates. When the distribution of the latent variable in our proposed model was symmetric, the correlation structures between recovery rates as well as between defaults were the same as those of Andersen and Sidenius (2004) in the sense of Kendall’s tau. We calibrated the model parameters by using the default probabilities and the joint default probabilities, which can be estimated from the historical data.

To estimate the portfolio loss probability, we proposed two efficient algorithms: conditional Monte Carlo (condMC) and its variant using the cross-entropy method (condMC-CE). These algorithms are modified versions of those proposed by Chan and Kroese (2010), who show that these two numerical methods are remarkably more efficient than the plain Monte Carlo simulation. We showed the relative efficiency of the algorithms by investigating the tail property of the model, which justifies the use of the proposed method. Via a simulation study, we investigated the performance of the proposed estimation algorithms. The condMC-CE is between 4 and 100 times more efficient than condMC in terms of variance reduction.

DECLARATION OF INTEREST

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REFERENCES


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Primary-firm-driven portfolio loss

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ABSTRACT

Many financial institutions provide loans to secondary firms, whose economic survival depends on the economic condition of primary firms. Even if loans from primary firms are not held in the loan portfolio, the financial distress of primary firms can adversely affect the loan portfolio of a financial institution. This paper describes a simple model that can be used for risk management. Our model directly incorporates the dependence of the conditional probability of default and loss given default of secondary firms on primary firms. Two simple examples show that failure to account for such dependence can result in the value-at-risk and the expected shortfall being greatly underestimated.

Keywords: primary and secondary firms; Gaussian latent-factor model; expected loss; value-at-risk (VaR); expected shortfall (ES).

1 INTRODUCTION

There are many industries in which a number of primary firms employ the services of secondary firms. The main source of revenue for many of these secondary firms is provided by one or more of these primary firms. Consequently, if a primary firm defaults, the economic consequences for secondary firms with whom it has economic relations can be severe; in extreme cases, it can cause default. In 2002, when Kmart defaulted, Fleming Companies, Inc., which supplied Kmart with all of its groceries,
was forced into liquidation (Turner 2006). After General Motors and Chrysler went bankrupt in 2009 many small suppliers went into liquidation and many employees lost their jobs (Beene 2009; Goolsbee and Krueger 2015). Jorion and Zhang (2009) document empirical evidence of the detrimental effects caused by the default of a primary firm. For banks providing loans to firms within such industries, failure to account for the economic structure within the industry may result in underestimation of the credit risk in the loan portfolio, as the presence of secondary firms in a loan portfolio will affect risk metrics such as value-at-risk (VaR) and expected shortfall (ES).

Many employees of secondary firms have personal loans, such as home and auto loans, with their local banks. If a primary firm suffers economic distress, such employees may be laid off or face reductions in the number of hours they are allowed to work, adversely affecting their creditworthiness. This may affect the credit risk of the financial institutions providing the loans to these employees. Financial institutions catering to primary and secondary firms and their employees need to recognize the risk arising from the dependence of secondary firms on primary firms. Many small financial institutions and online firms offering both corporate loans and personal loans have limited risk management facilities, and consequently there is a need for a simple model to incorporate this form of risk.

This paper examines how the presence of a primary firm affects the loss distribution of a portfolio of loans to secondary firms. The creditworthiness of a primary firm will affect the probability of default for secondary firms. Further, if a primary firm defaults, this will, in general, affect the distribution of the loss given default (LGD) of secondary firms. These two effects will adversely affect the VaR and ES. We demonstrate that both effects can be substantial. Many credit card holders may have either direct or indirect exposure to a primary firm. Given that the number of exposed cardholders can be large, this suggests we consider a large homogeneous portfolio case. We derive the conditional asymptotic distribution, drawing on the work of Gordy (2003).

Jarrow and Yu (2001) consider the consequences for counterparty risk of a secondary firm in the presence of a primary firm. They extend the reduced-form model introduced by Jarrow and Turnbull (1992, 1995) to consider obligors that have correlated defaults because of dependence on economic factors and economic relations. For primary firms, default is driven by economic factors. If a primary firm defaults, this will have an adverse effect on the creditworthiness of secondary firms. However, if a secondary firm defaults, it will have no impact on the creditworthiness of a primary firm. This type of model has been applied by Leung and Kwok (2005) in pricing credit default swaps subject to counterparty risk.

A similar type of problem arises in the pricing of insured debt. Heitfield and Barger (2003), in a discrete time framework, examined the issue of insured debt payments and the implications for regulatory capital. We extend their model to incorporate the impact of failure of a primary firm on both the probability of default and the LGD. The
probability of default and the LGD vary with the state of the economy (Frye 2000; Pykhtin 2003; Altman et al 2005; Acharya et al 2007; Chava et al 2011). We allow both the probability of default and the LGD to vary with the state of the economy and the creditworthiness of the primary firm, implying that defaults and the LGD will be correlated across firms.

Section 2 of this paper describes the model for the portfolio loss distribution. Section 3 considers a large homogeneous portfolio analysis and develops a relatively easy-to-use result for the loss distribution. Section 4 examines the practical implications and demonstrates that failure to account for the presence of a primary firm can result in substantial underestimation of the VaR and ES. Section 5 offers some brief concluding remarks.

2 MODEL DEVELOPMENT

We start by considering two obligors: a primary obligor, denoted by A, and a secondary obligor, S (using the terminology introduced by Jarrow and Yu (2001)). The probability of default and the LGD are both affected by the economy. The starting point for our model is similar to the model described by Heitfield and Barger (2003), who consider guaranteed debt. We extend their model to incorporate stochastic LGD. If A defaults, this can have an adverse effect on S; in the extreme case, S defaults. However, S can default without affecting A, as mentioned above. There are four possible states describing the joint default status of A and S. We adapt the Gaussian latent-factor model, similar in spirit to Pykhtin (2003), to describe the probability of default over a fixed horizon in the different states and the LGD. For a fixed time horizon, let $D_A$ denote the event of default for the primary firm, and let $D_A^N$ denote the event of no default. Similar notation is employed for the secondary firm.

For the primary firm, the probability of default is described by a Gaussian latent-factor model described by

$$X_A = \beta_A Z + \sqrt{1 - \beta_A^2} e_A,$$

(2.1)

where term $Z$ is common to all obligors, $e_A$ is purely idiosyncratic and $Z$ and $e_A$ are independent and identically distributed, zero-mean, unit-variance, normally distributed random variables. The term $\beta_A$ is constant, and $|\beta_A| \leq 1$. The probability of default is given by

$$P[D_A] = P[X_A \leq C_A] = \Phi(C_A),$$

(2.2)

where $\Phi(\cdot)$ is the cumulative normal distribution function and $C_A$ is the threshold, assumed known.

For secondary firms the probability of default will depend on the state of the economy and the creditworthiness of the primary firm, as reflected by $X_A$. Since $X_A$ is a
linear combination of the common factor $Z$ and an idiosyncratic component $e_A$, we write

$$X_S = \beta_S Z + \gamma_S e_A + \sqrt{1 - \beta^2_S - \gamma^2_S} \tilde{e}_S.$$ \hspace{1cm} (2.3)

where $\tilde{e}_S$ is an independent, purely idiosyncratic normally distribution random variable, with a zero mean unit variance. The coefficient $\gamma_S$ is nonnegative and $\beta^2_S + \gamma^2_S \leq 1$. Note that the joint distribution for $X_A$ and $X_S$ is bivariate normal, with correlation coefficient

$$\rho = \beta_A \beta_S + \gamma_S \sqrt{1 - \beta^2_A}.$$ \hspace{1cm} (2.4)

The probability of the secondary firm defaulting conditional on no default by the primary firm is

$$P[D_S \mid D^N_A] = P[X_S \leq C_S \mid D^N_A] = \frac{\Phi_2(C_S, -C_A; -\rho)}{1 - \Phi(C_A)},$$ \hspace{1cm} (2.5)

where $\Phi_2(\cdot, \cdot)$ is the cumulative bivariate normal distribution function.

If the primary firm has defaulted, we assume that it continues in existence as it tries to reorganize. The default will adversely affect the creditworthiness of the secondary firm: it will affect the default threshold and the structural specification of the latent variable $X_S$. The exact specification of the new threshold for the secondary firm is far from trivial. Whether the default of the primary firm is caused by firm-specific factors or economywide malaise will have a differing impact on secondary firms. Some secondary firms will be more diversified and less dependent on the primary firm than others. In each case the loan officer must decide how the different secondary firms will be affected. Here, we assume that default by the primary firm, for whatever reason, will increase the default threshold (Ebert and Lütkebohmert 2012). The new threshold is denoted by $\tilde{C}_S$. In the limit, if $\tilde{C}_S \to \infty$, default by the secondary firm becomes certain.

The relationship of the latent variable to the common factor $Z$ will, in general, change, and (2.3) is assumed to be

$$X_S = \tilde{\beta}_S Z + \tilde{\gamma}_S e_A + \sqrt{1 - \tilde{\beta}^2_S - \tilde{\gamma}^2_S} \tilde{e}_S.$$ \hspace{1cm} (2.6)

Note that the distribution for $X_A$ and $X_S$ is bivariate normal, with correlation coefficient

$$\tilde{\rho} = \beta_A \tilde{\beta}_S + \tilde{\gamma}_S \sqrt{1 - \beta^2_A}.$$ \hspace{1cm} (2.7)

The issue facing the loan office is specifying the new values for the coefficients. In summary, (2.3) and (2.6) can be written

$$X_S = \begin{cases} 
\beta_S Z + \gamma_S e_A + \sqrt{1 - \beta^2_S - \gamma^2_S} e_S & \text{if } X_A > C_A, \\
\tilde{\beta}_S Z + \tilde{\gamma}_S e_A + \sqrt{1 - \tilde{\beta}^2_S - \tilde{\gamma}^2_S} \tilde{e}_S & \text{if } X_A \leq C_A.
\end{cases}$$ \hspace{1cm} (2.8)
The probability of default by the secondary firm, conditional on default by the primary firm A, is given by

$$P[D_S \mid D_A] = P[X_S \leq \tilde{C}_S \mid D_A] = \frac{\Phi_2(\tilde{C}_S, C_A; \tilde{\rho})}{\Phi(C_A)}.$$

The unconditional probability of default for the secondary firm is given by

$$P[D_S] = P[D_S \mid D_A^N] P[D_A^N] + P[D_S \mid D_A] P[D_A]$$

$$= \Phi_2(C_S, -C_A; -\rho) + \Phi_2(\tilde{C}_S, C_A; \tilde{\rho}).$$

The first term on the right-hand side of the above expression is the probability of a secondary firm defaulting and no default by the primary firm; the second term is the probability of default by the secondary firm and default by the primary firm.

The LGD for the secondary firm is in general affected by the default of the primary firm. If the primary firm has not defaulted, then the LGD per unit of notional is denoted by $L_S$; if the primary firm has defaulted, it is denoted by $\tilde{L}_S$. This is summarized by

$$L = \begin{cases} L_S & \text{no default by primary firm,} \\ \tilde{L}_S & \text{default by primary firm.} \end{cases}$$

If we assume that the LGD is a constant varying only with the default status of the primary firm, then the expected loss due to default by a secondary firm is given by

$$E[L] = L_S P[D_S, D_A^N] + \tilde{L}_S P[D_S, D_A]$$

$$= L_S \Phi_2(C_S, -C_A; -\rho) + \tilde{L}_S \Phi_2(\tilde{C}_S, C_A; \tilde{\rho})$$

using (2.9). The first term on the right-hand side is the expected loss arising from default by a secondary firm and no default by the primary firm; the second term is the expected loss arising from default by a secondary firm and the primary firm.

We relax the assumption that the $L_S$ and $\tilde{L}_S$ are deterministic. The LGD is defined between 0 and 1. While a beta distribution is often assumed (Gupton and Stein 2002), it is well known that the LGD tends to increase as the probability of default increases. Paragraph 468 of the Basel Committee on Banking Supervision framework (Basel Committee on Banking Supervision 2004) requires that the LGD reflects the economic conditions at the time of default (see Basel Committee on Banking Supervision (2005) for guidance on Paragraph 468). Many extant studies assume that recovery rates are dependent on economic and firm-specific covariates. In Varma and Cantor (2005) and Acharya et al (2007) the losses are assumed to be unbounded. Pykhtin (2003) truncates the loss at unity. Schönbucher (2003) and Düllmann and Trapp (2004) assume the loss is described by a logit function, while Andersen and Sidenius (2005) assume a probit function. Chava et al (2011) find that the logit and probit models are quite similar.
in out-of-sample performance. Here, we follow Andersen and Sidenius (2005); the
LGD is described by a probit function:

\[ \text{LGD} = s[1 - \Phi(\mu + bZ + \sigma \xi)], \]  

(2.12)

where \( \xi \) is a zero-mean, unit-variance, normally distributed random variable, indepen-
dent of the common term \( Z \). The term \( s \) \((s \leq 1)\) represents the maximum LGD, and \( \mu \) 
and \( b \) are constants. The LGD is bounded \((0 \leq L \leq s)\). The loan officer must specify 
the value of \( \mu \), the dependence, \( b \), on the common factor, the idiosyncratic volatility, 
\( \sigma \), and the maximum LGD, \( s \). If data is available, the parameters can be estimated 
to the beta distribution, the shape of the density function can vary greatly, depending 
on the values of \( \mu \) and \( \sigma_L^2 = b^2 + \sigma^2 \), as shown in Andersen and Sidenius (2005).

The expected value of (2.12) is given by

\[ E[s[1 - \Phi(\mu + bZ + \sigma \xi)]] = s \left[ 1 - \Phi \left( \frac{\mu}{\sqrt{1 + \sigma_L^2}} \right) \right], \]  

(2.13)

as shown in Andersen and Sidenius (2005).

If there is no default by the primary firm, we assume the LGD for the secondary 
firm is given by

\[ L_S = s[1 - \Phi(\mu + bZ + \sigma \xi)]. \]  

(2.14)

The LGD and the probability of default are linked by the common factor, \( Z \). Let 
\( Y = \mu + bZ + \sigma \xi \), which is normally distributed with mean \( \mu \) and variance \( \sigma_L^2 = b^2 + \sigma^2 \). The latent factor, \( X_S \), and \( Y \) are bivariate normal with correlation coefficient 
\( \rho = \beta_S b/\sigma_L \). If the primary firm has defaulted, then the LGD for the secondary firm 
\[ \tilde{L}_S = \tilde{s}[1 - \Phi(\tilde{\mu} + \tilde{b}Z + \tilde{\sigma} \tilde{\xi})]. \]  

(2.15)

Let \( \tilde{Y} = \tilde{\mu} + \tilde{b}Z + \tilde{\sigma} \tilde{\xi} \). This is normally distributed with mean \( \tilde{\mu} \) and variance 
\( \tilde{\sigma}_L^2 = \tilde{b}^2 + \tilde{\sigma}^2 \). The latent factors \( X_S \) and \( \tilde{Y} \) are bivariate normal with correlation 
coefficient \( \tilde{\rho} = \beta_S \tilde{b}/\tilde{\sigma}_L \).

The LGD is only realized if the secondary firm defaults. To calculate expected LGD, 
we must consider how the primary firm will affect both the probability of default of 
the secondary firm and the LGD. The expected LGD is given by

\[ E[L \mid D_S] = E[s(1 - \Phi(\mu + bZ + \sigma \xi)) \mid 1_{X_S \leq c_S} 1_{X_A > c_A}] \]
\[ + E[\tilde{s}(1 - \Phi(\tilde{\mu} + \tilde{b}Z + \tilde{\sigma} \tilde{\xi})) \mid 1_{X_S \leq \tilde{c}_S} 1_{X_A \leq c_A}]. \]  

(2.16)

The first term on the right-hand side is the expected loss conditional on default by 
the secondary firm and no default by the primary firm, and the second term is the
expected loss conditional on default by the secondary firm and default by the primary firm. The expected loss is given by

\[ E[L] = E[s \Phi(-\mu - bZ - \sigma \xi)^1_{X_S \leq C_S} 1_{X_A > C_A}] + E[\tilde{s} \Phi(-\tilde{\mu} - \tilde{b}Z - \tilde{\sigma} \tilde{\xi})^1_{X_S \leq \tilde{C}_S} 1_{X_A \leq C_A}], \]  

(2.17)

which can be written in the form

\[ E[L] = s \int_{-\infty}^{\infty} \phi(z) \Phi\left(\frac{-\mu - bz}{\sqrt{1 + \sigma^2}}\right) \Phi_2\left(\frac{C_S - \beta_S z}{\sqrt{1 - \beta_S^2}}, \frac{-C_A + \beta_A z}{\sqrt{1 - \beta_A^2}}; \frac{-\gamma}{\sqrt{1 - \beta_S^2}}\right) dz \]

\[ + \tilde{s} \int_{-\infty}^{\infty} \phi(z) \Phi\left(\frac{-\tilde{\mu} - \tilde{b}z}{\sqrt{1 + \tilde{\sigma}^2}}\right) \Phi_2\left(\frac{\tilde{C}_S - \tilde{\beta}_S z}{\sqrt{1 - \tilde{\beta}_S^2}}, \frac{C_A - \beta_A z}{\sqrt{1 - \beta_A^2}}; \frac{\gamma}{\sqrt{1 - \beta_S^2}}\right) dz \]

(see Appendix A for the derivation). This expression can be evaluated using numerical integration. Alternatively, the integrals can be written in terms of trivariate normal cumulative distribution functions (see Appendix B for details).

### 3 LARGE HOMOGENEOUS PORTFOLIOS

Consider a portfolio containing loans to \( n \) secondary firms over a specified horizon. The notional of each loan is denoted by \( F_j, j = 1, \ldots, n \), and the total notional value of the portfolio is \( \sum_{j=1}^{n} F_j \). Two simplifying assumptions are made. First, there is only one primary firm, not held in the portfolio, that affects the secondary firms and, second, all loans are statistically identical. The portfolio loss per unit of notional is

\[ L_p = L_p^{(1)} + L_p^{(2)}, \]  

(3.1)

where

\[ L_p^{(1)} = \frac{1}{n} \sum_{j=1}^{n} L_j 1_{(X_{j,S} \leq C_S)} 1_{(X_A > C_A)}, \]

\[ L_p^{(2)} = \frac{1}{n} \sum_{j=1}^{n} \tilde{L}_j 1_{(X_{j,S} \leq \tilde{C}_S)} 1_{(X_A \leq C_A)}. \]

The term \( L_p^{(1)} \) is the loss if secondary firms default and the primary firm does not default, and \( L_j \) is the LGD of the \( j \)th secondary firm. The term \( L_p^{(2)} \) is the loss if the primary firm and secondary firms default, and \( \tilde{L}_j \) is the LGD of the \( j \)th secondary firm. Note that both the probability of default and the LGD are affected by the default of the primary firm. As \( L_j, j = 1, \ldots, N \), is defined over the unit interval, its variance is bounded.
Consider the first term on the right-hand side of (3.1). The expected value, conditional on \( Z \) and \( e_A \), is given by

\[
E[L_p^{(1)} \mid Z, e_A] = \frac{1}{n} \sum_{j=1}^{n} E(L_j \mid Z, e_A) E(1(X_{j,s} \leq C_s) \mid Z, e_A) 1(x_A > C_A)
\]

\[
\equiv h(Z, e_A) 1(x_A > C_A)
\]

(3.2)

using the conditional independence. Drawing on the work of Gordy (2003), we now show that the conditional variance goes to zero as the size of the portfolio increases. First, given conditional independence, we have

\[
\text{var}(L_p^{(1)} \mid Z, e_A) = \frac{1}{n^2} \sum_{j=1}^{n} \text{var}(L_j 1(X_{j,s} \leq C_s) \mid Z, e_A) 1_{C_A},
\]

recognizing that \( 1^2_j = 1_j \). Consider

\[
\text{var}(L_j 1(X_{j,s} \leq C_s) \mid Z, e_A) = E[(L_j)^2 \mid Z, e_A] E[(1(X_{j,s} \leq C_s))^2 \mid Z, e_A]
\]

\[
- [E(L_j \mid Z, e_A)]^2 E(1(X_{j,s} \leq C_s) \mid Z, e_A)^2
\]

\[
= \text{var}(L_j \mid Z, e_A) E[(1(X_{j,s} \leq C_s))^2 \mid Z, e_A]
\]

\[
+ \text{var}(1(X_{j,s} \leq C_s) \mid Z, e_A) [E(L_j \mid Z, e_A)]^2.
\]

The first equality follows given conditional independence, and the second follows after simplification. Therefore,

\[
\lim_{n \to \infty} \text{var}(L_p^{(1)} \mid Z, e_A) = 0,
\]

and, using Chebyshev’s inequality, conditional on \( (Z, e_A) \), \( L_p^{(1)} \) converges in probability to \( h(Z, e_A) \).

Similar analysis and assumptions are applied to \( L_p^{(2)} \), again conditional on \( Z \) and \( e_A \), so that

\[
E[L_p^{(2)} \mid Z, e_A] = \frac{1}{n} \sum_{j=1}^{n} E(\tilde{L}_j \mid Z, e_A) E[1(X_{j,s} \leq \tilde{C}_s) \mid Z, e_A] 1(x_A \leq C_A)
\]

\[
\equiv \tilde{h}(Z, e_A) 1(x_A \leq C_A),
\]

and \( L_p^{(2)} \) converges in probability to \( \tilde{h}(Z, e_A) \).

Given the assumption that all the firms are statistically identical, we have

\[
E[1(X_{j} \leq C_s) \mid Z, e_A] = \Phi\left( \frac{C_s - \beta_S Z - \gamma e_A}{\sqrt{1 - \beta_S^2 - \gamma^2}} \right)
\]
and
\[ E(L_j \mid Z, e_\Lambda) = s \left[ 1 - \Phi \left( \frac{\mu + bZ}{\sqrt{1 + \sigma^2}} \right) \right], \]
so that
\[ h(Z, e_\Lambda) = s \left[ 1 - \Phi \left( \frac{\mu + bZ}{\sqrt{1 + \sigma^2}} \right) \right] \phi \left( \frac{C_S - \beta S Z - \gamma e_\Lambda}{\sqrt{1 - \beta_S^2 - \gamma^2}} \right) \tag{3.3} \]
and
\[ \tilde{h}(Z, e_\Lambda) = \tilde{s} \left[ 1 - \Phi \left( \frac{\tilde{\mu} + \tilde{b}Z}{\sqrt{1 + \tilde{\sigma}^2}} \right) \right] \phi \left( \frac{\tilde{C}_S - \tilde{\beta}_S Z - \tilde{\gamma} e_\Lambda}{\sqrt{1 - \tilde{\beta}_S^2 - \tilde{\gamma}^2}} \right). \tag{3.4} \]

The loss distribution is given by
\[
P[L_P \leq y] = P[h(Z, e_\Lambda) 1_{(X_\Lambda > C_\Lambda)} + \tilde{h}(Z, e_\Lambda) 1_{(X_\Lambda \leq C_\Lambda)} \leq y] \\
= P[h(Z, e_\Lambda) + (\tilde{h}(Z, e_\Lambda) - h(Z, e_\Lambda)) 1_{(X_\Lambda \leq C_\Lambda)} \leq y]. \tag{3.5}
\]

Expression (3.5) does not in general lend itself to simplification. We consider a special case. Assume that default by the primary firm does not affect the LGD and the critical default barrier for the secondary firms (\(\tilde{C}_S = C_S\)), implying \(\tilde{h}(Z, e_\Lambda) = h(Z, e_\Lambda)\). This case is still of interest. First, the LGD and the probability of default for the primary and secondary firms are related through their common dependence on the economic factor \(Z\). Second, the idiosyncratic risk of the primary firm affects the probability of default of secondary firms, assuming the gamma coefficient, \(\gamma\), is positive. In this case, (3.5) simplifies to
\[
P[L_P \leq y] = P[h(Z, e_\Lambda) \leq y].
\]

Let
\[
a(Z) = \left\{ s \left[ 1 - \Phi \left( \frac{\mu + bZ}{\sqrt{1 + \sigma^2}} \right) \right] \right\}^{-1} > 0,
\]
so that
\[
P[L_P \leq y \mid Z] = P \left[ \phi \left( \frac{C_S - \beta S Z - \gamma e_\Lambda}{\sqrt{1 - \beta_S^2 - \gamma^2}} \right) \leq a(Z)y \mid Z \right].
\]

Now,
\[
C_S - \beta S Z - \sqrt{1 - \beta_S^2 - \gamma^2} \Phi^{-1}(a(Z)y) \leq \gamma e_\Lambda.
\]

Let
\[
g(Z; y) = \frac{C_S - \beta S Z - \sqrt{1 - \beta_S^2 - \gamma^2} \Phi^{-1}(a(Z)y)}{\gamma},
\]
so that
\[
P[L_P \leq y \mid Z] = P[g(Z; y) \leq e_\Lambda \mid Z] = \Phi(-g(Z; y))
\]
and
\[ P[L_P \leq y] = \int_{-\infty}^{\infty} \phi(z)\Phi(-g(z; y)) \, dz, \tag{3.6} \]
which can be evaluated using numerical integration. If \( \gamma \) equals zero, then the above expression simplifies to the result in Andersen and Sidenius (2005, Proposition 5).

4 PRACTICAL IMPLICATIONS

The risk characteristics of a loan portfolio will depend on the size of each loan, the number of primary firms, \( N_P \), the number of secondary firms, \( n_S \), and the number of firms that are not affected by primary firms, \( n_{\text{S}} \). We call these firms nonsecondary firms. We consider a loan portfolio with secondary and nonsecondary firms. There is one primary firm, not held in the loan portfolio, that can affect the secondary firms. The loss due to default can be represented by
\[ L_I = \sum_{k=1}^{n_S} F_k L_k 1(X_k \leq C_k) + \sum_{i=1}^{\tilde{n}_S} F_i L_i 1(X_i \leq C_i). \tag{4.1} \]
The two terms on the right-hand side represent the loss due to default of secondary firms and nonsecondary firms, respectively, in the loan portfolio, and \( F_k \) and \( F_i \) denote the notionals of the respective loans. Note that while the primary firm is not held in the loan portfolio, its presence can still have an adverse effect on the portfolio.

4.1 Example

To provide a benchmark, we start by considering a portfolio of nonsecondary firms, assuming the LGD is a constant. We then extend this analysis by introducing secondary firms. Finally, we consider the impact when the LGD is stochastic and correlated with the probability of default.

4.1.1 Benchmark

To provide a benchmark, we consider a portfolio of 100 nonsecondary firms. There are no primary and secondary firms, and the LGD is assumed to be known. The probability of default for each firm is assumed to be 2%; the face value of debt is 100 for all firms and the LGD is assumed known at 0.50 per dollar of notional. We start by assuming firms are independent. In (2.1) the beta coefficient is zero (\( \beta = 0 \)), implying no dependence on the common factor, \( Z \). The portfolio loss is described by a binomial distribution. Therefore, the expected loss is 100 and the standard deviation of the portfolio loss is 70. For the \( \beta = 1 \) case, default is driven solely by \( Z \). There are only two states (no defaults, or all firms default), and the distribution becomes a Bernoulli distribution. The expected loss is independent of \( \beta \) (see (2.2)) and remains
### TABLE 1  Expected portfolio loss.

<table>
<thead>
<tr>
<th>Beta</th>
<th>0.0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: Expected portfolio loss</td>
<td>99.95</td>
<td>100.17</td>
<td>99.34</td>
<td>99.76</td>
</tr>
<tr>
<td>VaR</td>
<td>287</td>
<td>267</td>
<td>487</td>
<td>1066</td>
</tr>
<tr>
<td>ES</td>
<td>325</td>
<td>316</td>
<td>643</td>
<td>1417</td>
</tr>
<tr>
<td>Case 2: Expected portfolio loss</td>
<td>103.73</td>
<td>104.31</td>
<td>104.34</td>
<td>103.58</td>
</tr>
<tr>
<td>VaR</td>
<td>351</td>
<td>402</td>
<td>570</td>
<td>1091</td>
</tr>
<tr>
<td>ES</td>
<td>468</td>
<td>507</td>
<td>732</td>
<td>1467</td>
</tr>
<tr>
<td>Case 3: Expected portfolio loss</td>
<td>110.93</td>
<td>112.67</td>
<td>113.27</td>
<td>112.38</td>
</tr>
<tr>
<td>VaR</td>
<td>818</td>
<td>949</td>
<td>1105</td>
<td>1367</td>
</tr>
<tr>
<td>ES</td>
<td>1082</td>
<td>1156</td>
<td>1336</td>
<td>1743</td>
</tr>
<tr>
<td>Case 4: Expected portfolio loss</td>
<td>104.12</td>
<td>105.67</td>
<td>108.05</td>
<td>110.33</td>
</tr>
<tr>
<td>VaR</td>
<td>319</td>
<td>410</td>
<td>655</td>
<td>1238</td>
</tr>
<tr>
<td>ES</td>
<td>411</td>
<td>549</td>
<td>905</td>
<td>1621</td>
</tr>
</tbody>
</table>

The beta coefficient for the primary firm is fixed at 0.50. The number in parentheses is the standard error of the estimate of the expected loss. The confidence interval is set at 99%. An antithetic Monte Carlo simulation is used. Case 1: no primary firms and 100 nonsecondary firms. For these nonsecondary firms, probability of default (PD) is 0.02 and LGD is 0.50. The face value of debt is 100 for all firms. Case 2: one primary firm, not held in the portfolio, affects the secondary firms. For the primary firm, PD is 0.01 and its beta is 0.50. Ten secondary firms are affected by the primary firm and ninety nonsecondary firms are not affected by the primary firm. For the nonsecondary firms, PD is 0.02 and LGD is 0.50. If the primary firm defaults, PD for the secondary firms becomes 0.20 and LGD is 0.70. For cases 2 and 3, gamma equals 0.50. Case 3: one primary firm, not held in the portfolio, affects the secondary firms. Thirty secondary firms are affected by the primary firm and seventy nonsecondary firms are not affected by the primary firm. All other parameter values remain unchanged. Case 4 is similar to case 3, except gamma equals zero.
unchanged at 100. The standard deviation of the portfolio loss increases to 700. This increase in the standard deviation will increase both the VaR and ES. The simulation results are shown in case 1 in Table 1. The expected loss remains unchanged as the beta coefficient increases. However, both the VaR and ES increase as beta increases. This is to be expected, as the nature of the distribution changes.

In case 2, we assume there are ten secondary firms and ninety nonsecondary firms in the portfolio, so the total number of firms remains constant at 100. There is one primary firm, not held in the portfolio, that affects the ten secondary firms. The probability of default for the primary firm is 1%. Initially, the probability of default for secondary and nonsecondary firms is 2%. If the primary firm defaults, the probability of default of the secondary firms increases to 20%. We make the simplifying assumption that the event of default by the primary firm does not affect the coefficients in (2.3): the coefficients in (2.3) and (2.6) are identical. What do change are the threshold and the LGD (which increases from 0.50 to 0.70 per dollar of face value).

When the beta coefficient of the secondary and nonsecondary firms equals zero ($\beta = 0$), the expected loss increases to 103.73 compared with 99.95 in case 1, reflecting the impact that the presence of the primary firm has on the conditional probability of default and LGD for the secondary firms. In the remaining cases, the beta coefficients of the secondary firms and nonsecondary firms are assumed to be identical.

For positive beta, the probability of default for secondary firms and the expected loss are functions of $\beta$: see (2.9) and (2.11), respectively. This is not the case for nonsecondary firms; the probability of default and the expected loss are independent of beta. However, higher moments in the loss distribution are affected by beta, as case 1 demonstrates. It is observed that the expected loss increases and then decreases. This is explained by considering the impact on the expected loss as the beta of the secondary firm increases. The expected loss depends on two terms: the first term is the probability of the secondary firm defaulting and no default by the primary firm, and the second term is the probability of the secondary and primary firms defaulting (see (2.11)). As beta increases, the correlation between the latent factors $X_A$ and $X_S$ increases (see (2.4)). The first probability decreases as the correlation increases, while the second probability increases. Eventually, the magnitude of the increases in the second factor is offset by the decreases in the first factor, implying that the expected loss starts to decrease. This effect depends on the parameter values and the correlation. In case 4, the expected loss does not decrease. Both the VaR and ES increase as beta increases.

Case 3 is similar to case 2, except there are now thirty secondary firms and seventy nonsecondary firms. There is one primary firm, not held in the portfolio, that affects the thirty secondary firms. The results are similar to case 2, though, with 30% of the portfolio being affected by the primary firm; the effects are more clearly discernible. For $\beta = 0$, the expected loss is now 110.95, compared with 103.73 in case 2. Similarly
to case 2, the expected loss increases as beta increases and then starts to decrease. The VaR and ES increase relative to case 2.

Case 4 is similar to case 3, except gamma is now zero, implying that the idiosyncratic term of the primary firm does not affect the probability of default of secondary firms (see (2.3)). There are two notable effects. First, the expected loss is lower than in case 3. For example, when beta equals zero, expected loss is now 103.67 compared with 110.95 in case 3. Second, as beta increases, the expected loss increases, unlike in cases 2 and 3. The magnitude of the correlation is smaller than in cases 2 and 3 (see (2.4)). As the beta increases, the correlation increases. The first probability decreases as the correlation increases, while the second probability increases. However, in this case, the magnitude of the increases in the second factor is greater than the decreases in the first factor, implying that the expected loss does not decrease. Both the VaR and ES increase as beta increases.

4.1.2 Stochastic LGD

The effects of stochastic LGD are examined in Table 2. To provide a benchmark, similar to case 1 in Table 1, we consider a portfolio of 100 nonsecondary firms. The expected loss per unit of total notional can be written in the form

\[
\frac{1}{100} \sum_{j=1}^{100} sE[1_{(X_j \leq C_{NS})}] - sE[\Phi(\mu + b Z + \sigma \xi_j)1_{(X_j \leq C_{NS})}] = \Phi_2 \left( \frac{\mu}{\sqrt{1 + \sigma^2 + b^2}}, C_{NS}; \rho \right),
\]

(4.2)

The first term on the right-hand side inside the summation is equal to \( sP[X_j \leq C_{NS}] \) and is independent of beta. The second term arises because of the stochastic nature of the LGD. We can write the second term in the form

\[
sE[\Phi(\mu + b Z + \sigma \xi_j)1_{(X_j \leq C_{NS})}] = \Phi_2 \left( \frac{\mu}{\sqrt{1 + \sigma^2 + b^2}}, C_{NS}; \rho \right),
\]

(4.3)

where \( \rho = -\beta b/\sqrt{1 + \sigma^2 + b^2} \). The expected loss per unit of total notional can be rewritten in the form

\[
\frac{1}{100} \sum_{j=1}^{100} s \left[ \Phi(C_{NS}) - \Phi_2 \left( \frac{\mu}{\sqrt{1 + \sigma^2 + b^2}}, C_{NS}; \rho \right) \right].
\]

(4.4)

The proof is given in Appendix C.

To provide a reasonable benchmark, the drift term \( \mu \) in (4.4) must be defined. The LGD in case 1 in Table 1 is assumed to be 0.50 per unit of face value. We assume that \( s = 1, b = 0.10 \) and \( \sigma = 0.35 \) and, using (2.13), we set \( \rho \) such that

\[
L = s \left[ 1 - \Phi \left( \frac{\mu}{\sqrt{1 + b^2 + \sigma^2}} \right) \right] \Rightarrow \frac{\mu}{\sqrt{1 + b^2 + \sigma^2}} = \phi^{-1} \left( \frac{1 - L}{s} \right),
\]

(4.5)
TABLE 2  Stochastic loss given default, value-at-risk and expected shortfall.

<table>
<thead>
<tr>
<th>Beta</th>
<th>0.0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected portfolio loss</td>
<td>100.13</td>
<td>104.46</td>
<td>109.03</td>
<td>113.35</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.10)</td>
<td>(0.17)</td>
<td>(0.35)</td>
</tr>
<tr>
<td>VaR</td>
<td>299</td>
<td>310</td>
<td>596</td>
<td>1295</td>
</tr>
<tr>
<td>ES</td>
<td>350</td>
<td>367</td>
<td>799</td>
<td>1763</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Beta</th>
<th>0.0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected portfolio loss</td>
<td>103.97</td>
<td>109.02</td>
<td>113.56</td>
<td>117.97</td>
</tr>
<tr>
<td></td>
<td>(0.11)</td>
<td>(0.12)</td>
<td>(0.19)</td>
<td>(0.36)</td>
</tr>
<tr>
<td>VaR</td>
<td>368</td>
<td>434</td>
<td>669</td>
<td>1308</td>
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<tr>
<td>ES</td>
<td>480</td>
<td>548</td>
<td>875</td>
<td>1774</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Beta</th>
<th>0.0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected portfolio loss</td>
<td>111.41</td>
<td>117.42</td>
<td>123.00</td>
<td>125.37</td>
</tr>
<tr>
<td></td>
<td>(0.20)</td>
<td>(0.21)</td>
<td>(0.28)</td>
<td>(0.42)</td>
</tr>
<tr>
<td>VaR</td>
<td>847</td>
<td>1007</td>
<td>1184</td>
<td>1572</td>
</tr>
<tr>
<td>ES</td>
<td>1105</td>
<td>1216</td>
<td>1461</td>
<td>2013</td>
</tr>
</tbody>
</table>

The beta coefficient for the primary firm is fixed at 0.50. The number in parentheses is the standard error of the estimate of the expected loss. The confidence interval is set at 99%. An antithetic Monte Carlo simulation is used. The drift term \( \beta \) in (2.14) is set such that the expected value of (2.14) remains unchanged at 0.50 per unit of face value. We assume that \( \beta = 0.10 \) and \( \sigma = 0.35 \). The drift term \( \bar{\beta} \) in (2.15) is set such that the expected value of (2.15) remains unchanged at 0.70 per unit of face value. We assume that \( \bar{\beta} = 0.10 \) and \( \bar{\sigma} = 0.35 \).

Case 1: no primary firms and 100 nonsecondary firms. For these nonsecondary firms, probability of default (PD) is 0.02. The face value of debt is 100 for all firms. Case 2: one primary firm, not held in the portfolio, affects the secondary firms. Ten secondary firms are affected by the primary firm and ninety nonsecondary firms are not affected by the primary firm. For the primary firm, PD is 0.01, and its beta is 0.50. For the nonsecondary firms, PD is 0.02 and LGD is 0.50. For the secondary firms, PD given the primary has not defaulted is 0.02. If the primary firm defaults, PD for the secondary firms becomes 0.20. For cases 2 and 3, \( \gamma = 0.50 \). Case 3: one primary firm, not held in the portfolio, affects the secondary firms. Thirty secondary firms are affected by the primary firm and seventy nonsecondary firms are not affected by the primary firm. All other parameter values remain unchanged.

where \( L = 0.50 \). Therefore,

\[
E[\Phi(\mu + bZ + \sigma \xi_j)1_{(X_j \leq NS)}] = \Phi_2 \left( \Phi^{-1} \left( 1 - \frac{L}{s} \right), C_{NS}; \rho \right).
\]
In case 2 in Table 1, when the primary firm defaults, the LGD jumps from 0.50 to 0.70. For case 2 in Table 2, we assume $\tilde{s} = 1$, $\tilde{b} = 0.10$ and $\tilde{\sigma} = 0.35$, and we define $\tilde{\mu}$ for this case using (4.5), where $L = 0.70$. The probability density functions for the two cases are shown in Figure 1. For the first case, when $L = 0.50$, the distribution is symmetric. For the second case, when $L = 0.70$, there is a large probability mass to the right of the mean and a long tail to the left of the mean.

In case 1 in Table 2, there are 100 nonsecondary firms. When beta is zero the expected loss is 100.13, and when beta is equal to 0.75 this increases to 113.35. This differs from the results in case 1 in Table 1, where the expected loss was constant. The fact that the expected loss varies with beta is to be expected, given (4.4), where the correlation is proportional to beta. The VaR and ES increase due to the stochastic nature of the LGD.

Cases 2 and 3 of Table 2 are similar in specification to those in Table 1. When the beta of the secondary firms is zero, the results are very similar. This is to be expected, as the loss and probability of default are orthogonal. Also, we normalized the mean of the stochastic LGD, so that the expected loss is identical to the nonstochastic case. For positive beta, the results are different, however, as the probability of default and the LGD are correlated. First, the expected loss increases as the beta of the secondary firms increases, unlike the results in Table 1. Second, the VaR and ES values are larger in Table 2. Again, this is to be expected given the stochastic nature of the LGD.
5 CONCLUDING REMARKS

Many financial institutions cater to secondary firms and their employees. Even if loans from primary firms are not held in the loan portfolio, financial distress in primary firms can adversely affect the loan portfolio of a financial institution. In this paper, we described a simple model that can be used for risk management, which directly incorporates the dependence of secondary firms on primary firms. Two simple examples showed that failure to account for such dependence can result in the VaR and ES being greatly underestimated.

APPENDIX A. EXPECTED LOSS

Consider first the case when the primary firm has not defaulted. We need to evaluate

\[ A_1 \equiv E[\Phi(-\mu - b Z - \sigma \xi) 1_{X_S \leq C_S} 1_{X_A > C_A}] . \]

By conditioning on \( Z \) and \( e_A \), we can write

\[ A_1 \equiv E \{ E[\Phi(-\mu - b Z + \sigma \xi) 1_{X_S \leq C_S} | Z, e_A] 1_{X_A > C_A} \} \]
\[ = E \{ E[\Phi(-\mu - b Z + \sigma \xi) | Z, e_A] E[1_{X_S \leq C_S} | Z, e_A] 1_{X_A > C_A} \} . \]

The second line follows given conditional independence. Using the result in Andersen and Sidenius (2005, p. 65), we have

\[ E[\Phi(-\mu - b Z + \sigma \xi) | Z, e_A] = \Phi \left( \frac{-\mu - b Z}{\sqrt{1 + \frac{\sigma^2}{\mu^2}}} \right) . \]

Next, conditional on no default by the primary firm,

\[ E[1_{X_S \leq C_S} | Z, e_A] = \Phi \left( \frac{C_S - \beta_S Z - \gamma e_A}{\sqrt{1 - \beta_S^2 - \gamma^2}} \right) , \]

so that

\[ A_1 = E \left[ \Phi \left( \frac{-\mu - b Z}{\sqrt{1 + \frac{\sigma^2}{\mu^2}}} \right) \Phi \left( \frac{C_S - \beta_S Z - \gamma e_A}{\sqrt{1 - \beta_S^2 - \gamma^2}} \right) 1_{X_A > C_A} \right] \]
\[ = \int_{-\infty}^{\infty} \Phi(z) \Phi \left( \frac{-\mu - b Z}{\sqrt{1 + \frac{\sigma^2}{\mu^2}}} \right) \]
\[ \times \int_{(C_A-\beta_A z)/(1-\beta_A^2)}^{\infty} \Phi \left( \frac{C_S - \beta_S z - \gamma e_A}{\sqrt{1 - \beta_S^2 - \gamma^2}} \right) \phi(e_A) \, de_A \, dz \]
\[ = \int_{-\infty}^{\infty} \Phi(z) \Phi \left( \frac{-\mu - b Z}{\sqrt{1 + \frac{\sigma^2}{\mu^2}}} \right) \Phi_2 \left( \frac{C_S - \beta_S z - \gamma e_A}{\sqrt{1 - \beta_S^2 - \gamma^2}} ; -\frac{C_A + \beta_A z}{\sqrt{1 - \beta_A^2}} ; -\frac{\gamma}{\sqrt{1 - \beta_S^2 - \gamma^2}} \right) \, dz , \]

where \( \phi(\cdot) \) is the probability density function of a zero-mean, unit-variance, normally distributed random variable.
For the case when the primary firm has defaulted, we need to consider
\[
\Lambda_2 \equiv E[\Phi(-\bar{\mu} - \bar{b} Z + \bar{\xi})1_{X_S \leq \tilde{c}_S} 1_{X_A \leq \tilde{c}_A}].
\]

Hence,
\[
\Lambda_2 = \int_{-\infty}^{\infty} \phi(z) \Phi\left(\frac{-\bar{\mu} - \bar{b} z}{\sqrt{1 + \bar{\sigma}^2}}\right) \Phi_2\left(\frac{\tilde{c}_S - \tilde{\beta} S z}{\sqrt{1 - \tilde{\beta}_S^2}}, \frac{C_S - \beta S z}{\sqrt{1 - \beta_S^2}}, \frac{C_A - \beta A z}{\sqrt{1 - \beta_A^2}}; \frac{\gamma}{\sqrt{1 - \beta_A^2}}\right) dz.
\]

The expected loss is equal to
\[
s \int_{-\infty}^{\infty} \phi(z) \Phi\left(\frac{-\mu - b z}{\sqrt{1 + \sigma^2}}\right) \Phi_2\left(\frac{C_S - \beta S z}{\sqrt{1 - \beta_S^2}}, \frac{-C_A + \beta A z}{\sqrt{1 - \beta_A^2}}; \frac{\gamma}{\sqrt{1 - \beta_A^2}}\right) dz
\]
\[
+ \bar{s} \int_{-\infty}^{\infty} \phi(z) \Phi\left(\frac{-\mu - \bar{b} Z}{\sqrt{1 + \bar{\sigma}^2}}\right) \Phi_2\left(\frac{\tilde{c}_S - \tilde{\beta} S z}{\sqrt{1 - \tilde{\beta}_S^2}}, \frac{C_A - \beta A z}{\sqrt{1 - \beta_A^2}}; \frac{\gamma}{\sqrt{1 - \beta_A^2}}\right) dz.
\]

**APPENDIX B. ALTERNATIVE EXPRESSION FOR THE EXPECTED LOSS**

**Lemma B 1** We have
\[
\int_{-\infty}^{\infty} \phi(z) \Phi(a_1 + c_1 z) \Phi_2(a_2 + c_2 z, a_3 + c_3 z; \rho) dz
\]
\[
= \Phi_3\left(\frac{a_1}{\sqrt{1 + c_1^2}}, \frac{a_2}{\sqrt{1 + c_2^2}}, \frac{a_3}{\sqrt{1 + c_3^2}}; \Omega\right).
\]

The elements of the correlation matrix, \( \Omega \), are defined below.

**Proof** Let
\[
x_1 = -c_1 z + \delta_1,
\]
\[
x_2 = -c_2 z + \delta_2,
\]
\[
x_3 = -c_3 z + \frac{d_2 \delta_2 + \delta_3}{\sqrt{1 + d_2^2}},
\]
where \( \delta_j \) are independent and identically normally distributed, zero-mean, unit-variance random variables and independent of \( z \). Let
\[
\delta_4 = \frac{\delta_3 + d_2 \delta_2}{\sqrt{1 + d_2^2}} \sim N(0, 1) \quad \text{and} \quad \hat{x}_j \equiv \frac{x_j}{\sqrt{1 + c_j^2}} \sim N(0, 1), \quad j = 1, 2, 3.
\]
The correlation between $\delta_2$ and $\delta_4$ is denoted by $\rho(\delta_2, \delta_4) = d_2/\sqrt{1 + d_2^2} = \rho$. The correlation elements of $\Omega$ are given by

$$
\rho(\hat{x}_1, \hat{x}_2) = \frac{c_1c_2}{\sqrt{1 + c_1^2}\sqrt{1 + c_2^2}}, \\
\rho(\hat{x}_1, \hat{x}_3) = \frac{c_1c_3}{\sqrt{1 + c_1^2}\sqrt{1 + c_3^2}}, \\
\rho(\hat{x}_2, \hat{x}_3) = \frac{c_2c_3 + d_2/\sqrt{1 + d_2^2}}{\sqrt{1 + c_2^2}\sqrt{1 + c_3^2}}.
$$

Consider

$$
P(x_1 \leq a_1, x_2 \leq a_2, x_3 \leq a_3) = \Phi_3 \left( \frac{a_1}{\sqrt{1 + c_1^2}}, \frac{a_2}{\sqrt{1 + c_2^2}}, \frac{a_3}{\sqrt{1 + c_3^2}} ; \Omega \right). 
$$

(B.1)

Now,

$$
P(x_1 \leq a_1, x_2 \leq a_2, x_3 \leq a_3)
= E[P(x_1 \leq a_1 \mid z)P(x_2 \leq a_2, x_3 \leq a_3 \mid z)]
= E[P(\delta_1 \leq a_1 + c_1 z \mid z)P(\delta_2 \leq a_2 + c_2 z, \delta_4 \leq a_3 + c_3 z \mid z)]
= \int_{-\infty}^{\infty} \phi(z) \Phi(a_1 + c_1 z) \Phi(a_2 + c_2 z, a_3 + c_3 z; \rho) \, dz.
$$

(B.2)

The first equality follows because of conditional independence. Equating (B.1) and (B.2) gives the result.

\[ \square \]

**APPENDIX C. PROOF OF EXPRESSION (4.4)**

We want to evaluate

$$
E[\Phi(\mu + b Z + \sigma \xi)1(X_j < c_{NS})] = E\{E[\Phi(\mu + b Z + \sigma \xi)1(X_j < c_{NS}) \mid Z]\}
= E\left[ \Phi \left( \frac{\mu + b Z}{\sqrt{1 + \sigma^2}} \right) \Phi \left( \frac{C_{NS} - \beta Z}{\sqrt{1 - \beta^2}} \right) \right].
$$

**Lemma C1** We have

$$
\int_{-\infty}^{\infty} \phi(z) \Phi(a_1 + b_1 z) \Phi(a_2 + b_2 z) \, dz = \Phi_2 \left( \frac{a_1}{\sqrt{1 + b_1^2}}, \frac{a_2}{\sqrt{1 + b_2^2}} ; \rho \right).
$$

**Proof** Let $z_1 = -b_1 z + \delta_1$ and $z_2 = -b_2 z + \delta_2$, where $z$, $\delta_1$ and $\delta_2$ are zero-mean, unit-variance, independent normally distributed random variables. Now, $z_1$ and $z_2$ are bivariate normally distributed, with zero means, variances $1 + b_1^2$ and $1 + b_2^2$ and correlation $\rho = b_1 b_2/[(1 + b_1^2)(1 + b_2^2)]^{1/2}$. Therefore,

$$
P(z_1 \leq a_1, z_2 \leq a_2) = \Phi_2 \left( \frac{a_1}{\sqrt{1 + b_1^2}}, \frac{a_2}{\sqrt{1 + b_2^2}} ; \rho \right).
$$
Now,
\[
P(z_1 \leq a_1, z_2 \leq a_2) = E[P(z_1 \leq a_1, z_2 \leq a_2 | z)]
\]
\[
= E[P(z_1 \leq a_1 | z) P(z_2 \leq a_2 | z)]
\]
\[
= E[P(\delta_1 \leq a_1 + b_1 z | z) P(\delta_2 \leq a_2 + b_2 z | z)]
\]
\[
= \int_{-\infty}^{\infty} \phi(z) \Phi(a_1 + b_1 z) \Phi(a_2 + b_2 z) \ dz.
\]

By comparison with (4.3), we have
\[
a_1 = \frac{\mu}{\sqrt{1 + \sigma^2}}, \quad b_1 = \frac{b}{\sqrt{1 + \sigma^2}}
\]
and
\[
a_2 = \frac{C_{NS}}{\sqrt{1 - \beta^2}}, \quad b_2 = -\frac{\beta}{\sqrt{1 - \beta^2}}.
\]
implying that
\[
\frac{a_2}{\sqrt{1 + b_2^2}} = C_{NS} \quad \text{and} \quad \rho = -\beta \frac{b_1}{\sqrt{1 + b_1^2}}.
\]
Finally,
\[
E\{s[1 - \Phi(\mu + b Z + \sigma \xi_j)] 1_{X_j \leq C_{NS}}\} = s \left[ \Phi(C_{NS}) - \Phi_2\left(\frac{\mu}{\sqrt{1 + \sigma^2 + b^2}}, C_{NS}; \rho\right) \right].
\]

DECLARATION OF INTEREST

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REFERENCES


Research Paper

Adapting the Basel II advanced internal-ratings-based models for International Financial Reporting Standard 9

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ABSTRACT

Banks around the globe are implementing International Financial Reporting Standard 9 (IFRS 9), which is a considerable effort. A key element of IFRS 9 is a forward-looking “expected loss” impairment model, which is a significant shift from the incurred-loss model. We examine how we may use advanced internal-ratings-based (A-IRB) models in the estimation of expected credit losses for IFRS 9 purposes. We highlight the necessary model adaptations required to satisfy the new accounting standard. By leveraging on the A-IRB models, banks can lessen their modeling efforts in fulfilling IFRS 9 and capture the synergy between different modeling endeavors within institutions. In outlining the proposed probability of default, loss given default and exposure at default models, we provide detailed examples of how they may be implemented on secured lending. Moreover, in discussing the issues related to the estimation of the expected credit loss for IFRS 9, we highlight the challenges involved and propose practical solutions to deal with them. For instance, we propose the use of...
a convexity adjustment approach to circumvent the need for assigning probabilities in multiple-scenario analysis.

**Keywords**: advanced internal-ratings-based (A-IRB) approach; IFRS 9; probability of default (PD); loss given default (LGD); exposure at default (EaD); expected loss (EL).

## 1 INTRODUCTION

The recent global financial crisis highlighted the deficiency of the existing accounting standard, International Accounting Standard 39 (IAS 39), in the use of an incurred-loss model, which is deemed to be backward-looking, to account for credit losses on loans and other financial instruments (Ernst & Young 2014). In the pursuit of a more timely recognition of credit loss in financial statements, the International Accounting Standards Board (IASB) has introduced a forward-looking expected credit loss model in a new accounting standard, “IFRS 9 Financial Instruments” (IFRS 9), to be adopted not later than January 1, 2018.¹ Unlike in IAS 39, where credit losses are only recognized upon the occurrences of credit events, IFRS 9 requires lenders (or asset holders) to recognize expected credit loss over the life of financial instruments. Moreover, the expected credit losses are to be measured on either a (forward-looking) twelve-month or a lifetime basis, depending on whether there has been a material increase in credit risk since the initial recognition. With its forward-looking nature and its new “three-bucket” approach (see Section 2 for more details), not only will credit losses be recognized earlier but more losses will potentially be recognized. The implementation of IFRS 9 is expected to result in an increase in the overall credit loss allowances of many banks and to have important implications for the regulatory capital requirements of such financial institutions (Deloitte 2013; Ernst & Young 2014).

Due to the tight implementation deadline and potentially significant implications, banks around the globe are deploying many of their resources in developing the necessary forward-looking expected loss (EL) impairment model, which is a significant shift from the current incurred-loss model. There is a certain degree of subjectivity, and considerable judgement needs to be made in developing and implementing these models. Different modeling approaches have been proposed by the accounting community (see, for example, Global Public Policy Committee 2016).² An industry

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¹ IASB published the final version of IFRS 9 in July 2014. It replaced earlier versions of IFRS 9 on classification and measurement requirements (introduced in 2009 and 2010) and a hedge accounting model (introduced in 2013).

² Given the important implications of the new standard for the capital requirement of financial institutions, the Bank for International Settlements also provides supervisory guidance on the accounting for expected credit losses (Basel Committee on Banking Supervision 2015).
practice is yet to be established for the estimation of expected credit losses in satisfying IFRS 9.\textsuperscript{3} Given the similarities between the IFRS 9’s credit risk measure and that required to satisfy the Basel Committee on Banking Supervision’s (BCBS’s) regulatory requirement, a pragmatic solution is for banks to build on their internal models under the advanced internal-ratings-based (A-IRB) approach and leverage their well-established credit risk stress testing models to satisfy the IFRS 9 modeling needs. In this paper, we examine how we can utilize a suite of A-IRB models to estimate both one-year and lifetime expected credit losses for IFRS 9. Specifically, we adapt the A-IRB probability of default (PD), loss given default (LGD) and exposure at default (EaD) models for IFRS 9 use and show how we can arrive at the EL measure by integrating the PD, LGD and EaD parameters obtained from these models. To ensure the EL measure can reflect the current state of the economy and business cycle, the particular kind of model we examined is specifically developed to be dynamically driven by key macroeconomic variables. This kind of time series conditional model is commonly used to fulfill the stress testing requirement under Basel II (see, for example, Blümke 2010; Miu and Ozdemir 2009; Ozdemir and Miu 2008; Simons and Rolwes 2009; Yang and Du 2015). In this paper, we focus on the estimation of the expected credit loss for secured lending, which represents a significant part of the overall credit portfolio of a typical commercial bank.

In adapting the A-IRB models for IFRS 9 use, we need to be aware of a number of fundamental differences between the IFRS 9 and A-IRB parameters.

\begin{enumerate}
\item A-IRB parameters are estimated based on a one-year risk horizon (as the Basel II capital horizon is one year), whereas IFRS 9 parameters need to be estimated in intervals till the maturity of each product (ie, the whole term structure is needed). This is because under IFRS 9 both one-year EL and “lifetime” EL (that is, the EL estimated over the effective maturity of the product) are required.

\item The A-IRB PD parameter could be either conditional (typically known as point-in-time (PIT)) or unconditional (typically known as through-the-cycle (TTC)). The risk rating philosophy of a bank governs the choice between the two. Nevertheless, in most cases, banks use a hybrid PD philosophy with elements of both, but with a bias toward TTC (Miu and Ozdemir 2010). The A-IRB LGD and EaD parameters are typically TTC. Therefore, an EL estimate based on these A-IRB estimates would be a predominately TTC or “unconditional” estimate of the expected losses. IFRS 9, on the other hand, calls for a conditional EL that requires all PD, LGD and EaD parameters to be conditioned on the expected macroeconomic environment. Conditional PD estimations have
\end{enumerate}

\textsuperscript{3} McPhail and McPhail (2014) highlight the strengths and weaknesses of different modeling approaches that may be used to forecast credit losses.
commonly been used for stress testing under A-IRB. The use of conditional (ie, PIT) LGD and EaD estimates is, however, less common for A-IRB purposes (Ozdemir and Miu 2008).

(3) In general, A-IRB loss estimation is an “economic loss”, whereas IFRS 9 loss estimation is an “accounting loss”. There are several divergences in LGD estimations.

(a) The down-turn LGD adjustment used in A-IRB is not appropriate for IFRS 9, as it is an adjustment for the tail of the loss distribution to compensate for the fact that the Basel II formula ignores the correlations between the PD and LGD.

(b) Indirect recovery expenses (eg, overheads), which are included in A-IRB’s economic loss by being incorporated into LGD estimations, are supposed to be excluded from the IFRS 9 calculation, as they cannot be allocated directly to individual loans from an accounting standpoint.

(c) Another potential divergence will occur if IFRS 9’s accounting LGD will not use the same discount rate for recovery cashflows as used by the A-IRB’s economic LGD.

In this paper, we address different modeling issues related to (1) and (2) above. The “economic” versus “accounting” measurement issue in (3) is considered to be relatively less material, and we defer the detailed discussion of it to a future study.

This paper contributes to the practice of credit risk modeling by banks and financial institutions in adopting IFRS 9 by examining how we may use A-IRB models in the estimation of expected credit losses. In doing so, we highlight the importance of the necessary model adaptations required to satisfy the new accounting standard of impairment measurement. By leveraging on the A-IRB models, banks can lessen their modeling efforts in fulfilling IFRS 9, and capture the synergy between different modeling endeavors within the institutions. In outlining the proposed PD, LGD and EaD models, we provide detailed examples of how they may be implemented on secured lending. Moreover, in discussing the issues related to the estimation of the expected credit loss for IFRS 9, we highlight the challenges involved, and propose practical solutions to deal with them. For instance, one of the difficulties is in the calculation of the expected impairment loss through scenario analysis, which appears to be a common practice in forecasting the twelve-month or lifetime credit loss of an asset. Under this approach, we first estimate the losses under different plausible (economic) scenarios and then calculate the most likely value of the loss by evaluating a probability-weighted average value across the different scenarios. Assigning robust probabilities consistently to future scenarios is very difficult (if it is achievable at all).
The inherent subjectively will create material dispersion of results among financial institutions with similar underlying assets. It will therefore lead to difficulty in interpreting the results for the regulators and the investors alike. Dealing with multiple scenarios is also a considerable extra operational effort. In this paper, we propose a novel approach to deal with the issue without necessarily involving multiple scenarios, which can enhance the objectivity and replicability of the modeling results (see Section 2 for details).

We structure the paper as follows. In Section 2, we provide an overview of the task to calculate expected credit loss, and of the models that need to be developed under IFRS 9. We also introduce a mechanism to classify assets into the three risk buckets and discuss the implication of IFRS 9 on the procyclicality of credit provision. In Sections 3 and 4, respectively, we show how the conditional PD and LGD models for secured lending may be formulated based on their A-IRB counterparts. In Section 5, we examine the methodology for the estimation of EaD, together with the integration of the conditional parameters, to arrive at an EL measure for IFRS 9. We conclude with a few remarks in Section 6.

2 EXPECTED LOSS ASSESSMENT FRAMEWORK UNDER IFRS 9

2.1 Forward-looking expected loss impairment model

A key element of IFRS 9 is a forward-looking EL impairment model. The new standard requires that the EL estimate be forward-looking and incorporate available information at the time of estimation. Should a credit downturn (or upturn) be expected to “materially impact” the forward-looking credit quality of the obligors, “adjustments” to the (EL-based) estimates of the provisions are required.

IFRS 9 is not prescriptive about how exactly the changes in the credit/macro-economic environment should be reflected in the EL estimation. However, a replicable, transparent and defendable mechanism to translate the change in the credit environment to the change in the portfolio’s EL estimation is needed. Essentially, to satisfy IFRS 9, we are interested in calculating the EL of a credit facility, which can be defined as

\[ EL = E[\text{LGD} \times \text{EaD}] \times \text{PD}, \]  

(2.1)

where PD is probability of default, LGD is the random variable of loss given default and EaD is the random variable of exposure at default of the facility. The PD and the expectation are assessed based on the current forecast of the credit environment (CFCE). The definitions of PD, LGD and EaD under IFRS 9 are essentially the same as those under the Basel II A-IRB approach, perhaps with the minor exception that, in calculating LGD in IFRS 9, we need to exclude those indirect costs (eg, overheads) that cannot be attributed to individual facilities.
In practice, it is common to articulate the CFCE in terms of some kind of probability measure. Let us suppose the CFCE is characterized in terms of gross domestic product (GDP), and that we want to calculate the EL under the assumption that there are 35%, 50% and 15% chances that the GDP growth rate is 1.0%, 1.5% and 2.0%, respectively. Equation (2.1) can therefore be expressed as

$$\begin{align*}
\text{EL} &= E[\text{LGD} \times \text{EaD} | \text{GDP} = 1.0\%] \times \text{PD}(\text{GDP} = 1.0\%) \times 35\
&+ E[\text{LGD} \times \text{EaD} | \text{GDP} = 1.5\%] \times \text{PD}(\text{GDP} = 1.5\%) \times 50\
&+ E[\text{LGD} \times \text{EaD} | \text{GDP} = 2.0\%] \times \text{PD}(\text{GDP} = 2.0\%) \times 15, \quad (2.2)
\end{align*}$$

where $$E[\text{LGD} \times \text{EaD} | \text{GDP} = X\%]$$ denotes the expected value of LGD × EaD of the facility conditional on a GDP growth rate equal to X%, and PD(\text{GDP} = X\%) denotes the probability of default assessment of the facility conditional on a GDP growth rate equal to X%. Equation (2.2) is therefore an EL assessment according to the “expected” economic outlook at the time of estimation. If we take the usual approximation by assuming LGD and EaD are independent, (2.2) becomes

$$\begin{align*}
\text{EL} \cong E[\text{LGD} | \text{GDP} = 1.0\%] \times E[\text{EaD} | \text{GDP} = 1.0\%] \\
&\times \text{PD}(\text{GDP} = 1.0\%) \times 35\
&+ E[\text{LGD} | \text{GDP} = 1.5\%] \times E[\text{EaD} | \text{GDP} = 1.5\%] \\
&\times \text{PD}(\text{GDP} = 1.5\%) \times 50\
&+ E[\text{LGD} | \text{GDP} = 2.0\%] \times E[\text{EaD} | \text{GDP} = 2.0\%] \\
&\times \text{PD}(\text{GDP} = 2.0\%) \times 15. \quad (2.3)
\end{align*}$$

Banks have been conducting their stress testing exercise (eg, for Comprehensive Capital Analysis and Review (CCAR) and Internal Capital Adequacy Assessment Process (ICAAP) purposes under Basel II) by estimating conditional EL under a “stressed outlook” or, more precisely, under an economic outlook that corresponds to a selected “stress scenario”; eg, when the GDP growth rate equals −2.0%,

$$\begin{align*}
\text{EL} &= E[\text{LGD} \times \text{EaD} | \text{GDP} = -2.0\%] \times \text{PD}(\text{GDP} = -2.0\%) \\
&\cong E[\text{LGD} | \text{GDP} = -2.0\%] \times E[\text{EaD} | \text{GDP} = -2.0\%] \\
&\times \text{PD}(\text{GDP} = -2.0\%). \quad (2.4)
\end{align*}$$

Thus, both IFRS 9 and stress tests are conditional estimates, but what they are conditioned on is very different. Although IFRS 9 is not a stress testing exercise, banks can utilize their existing PD stress testing models for IFRS 9 purposes as long as they use the current forecast, as opposed to the stress scenario, as their input. Let us consider how we can operationalize this. Suppose we have a PD stress testing model developed for A-IRB purposes that allows us to calculate the PD conditional on the GDP growth
rate. Typically, such a model is forward-looking, relating the PD over the next year to today’s observed information (i.e., GDP), so that the PD can be used for the capital requirement calculation over a twelve-month risk horizon starting today.\footnote{It is quite likely that such a PD stress testing model is built on a statistical framework, where the point estimate of PD is subject to a certain degree of estimation errors. In our methodology, we abstract from such estimation errors.} Suppose the most recent GDP statistic (say GDP has grown by 1.7\%) was published two months ago, and negative information on the economy has been revealed during the previous two months. Given that this negative information has not yet been captured by the most recent GDP statistics, we need to come up with a “forecast” on the GDP growth rate that we believe can more accurately reflect the current state of the credit environment. Suppose, based on our assessment, a lower GDP growth rate forecast of 1.5\% is considered to be most probable.\footnote{How do we come up with the 1.5\% annual GDP growth rate “forecast” for today? One way is to first forecast the annual GDP growth rate to be realized at the next GDP publication date, which is ten months from now, given the current economic outlook. Suppose the forecast is 0.5\%. We then interpolate 1.7\% (the GDP growth rate two months ago) and 0.5\% (the GDP growth rate ten months from now) over time to obtain a “forecast” of 1.5\% for today.} We then use the GDP growth rate of 1.5\% as our stress testing model input to calculate the PD, and in turn the EL, over the next twelve months for IFRS 9 purposes. Besides the EL over the next twelve months, IFRS 9 also calls for the calculation of “lifetime” EL (more details provided below). To satisfy this objective, we need to adapt our existing stress testing models so they can be used to calculate PD over any twelve-month period in the future. Suppose we want to calculate the EL over a twelve-month period starting three months from now. Given the current economic environment, we need to first come up with a forecast of the GDP growth rate over the twelve-month period ending three months from now. We then input it into the stress testing model to obtain a PD corresponding to that twelve-month period, enabling us to calculate the respective EL.

There is one more practical issue we need to deal with. Note that in (2.3) it might not be good enough to calculate EL by simply evaluating the product of PD and the expected values of LGD and EaD under the “expected” economic condition (i.e., based on the expected GDP growth rate of \(1.0\% \times 35\% + 1.5\% \times 50\% + 2.0\% \times 15\% = 1.4\%\)). This is because \(PD(GDP) \neq PD(GDP = 1.4\%) \times 35\% + PD(GDP = 1.5\%) \times 50\% + PD(GDP = 2.0\%) \times 15\%\), (2.5a)
In other words, specifying only the expected economic condition of GDP = 1.4% is not sufficient to calculate EL. The CFCE can only be fully defined, and thus the EL ascertained, by also specifying the probabilities (i.e., 35%, 50% and 15%) of realizing all possible economic conditions (GDP = 1.0%, 1.5% and 2.0%). It could be tricky even to assign probability measures to near-term economic conditions, let alone assigning those for economic conditions to be realized more than a couple of years from now. In practice, the assignment of probability measures is quite likely to be ad hoc and subjective. Such probability measures are also difficult to estimate in a consistent fashion. The decision process is therefore not easy to replicate, and thus the outcomes are not easily defendable. In the online appendix, we outline a methodology to correct for the bias in the EL calculation as a result of the nonlinearity in the functional forms of PD(GDP), E[LGD | GDP] and E[EaD | GDP]. By doing so we can evaluate EL with only the point estimate of the expected economic condition and its standard deviation, which could be determined in a more objective way than the full probability measure.

### 2.2 EL buckets and their PD-based triggers

IFRS 9 requires the evaluation of different kinds of EL measures for loans classified into three different “buckets” of progressively higher loss potential. Specifically, the EL of loans in bucket 1 is estimated over a one-year horizon, whereas the EL of loans in bucket 2 is calculated over the remaining term to maturity (referred to as “life-time” EL). Finally, the EL of the impaired loans in bucket 3 is estimated based on the best estimates of recovery values.6 IFRS 9 calls for a replicable, transparent and defendable mechanism to move loans among the three buckets with respect to significant changes in the expected credit environment. Loans will be reclassified from one bucket to another if certain predefined triggers are activated. It is important to note that IFRS 9 requires these triggers be based on PD only. Specifically, the trigger is based on the PD under the CFCE. In terms of our above example, the PD

---

6 IFRS 9 also calls for lifetime EL for impaired loans in bucket 3. It can be argued that the lifetime of these impaired loans is shorter. However, a one-to-two-year workout period can be expected for some asset classes.
can be evaluated as

\[ E[PD \mid \text{under CFCE}] = PD(GDP = 1.0\%) \times 35\% + PD(GDP = 1.5\%) \times 50\% + PD(GDP = 2.0\%) \times 15\%. \]

It is not ideal to ignore any changes in LGD and EaD in devising the triggers. Suppose the change in the economic condition underlying the CFCE affects only LGD (e.g., the collateral value goes down materially) and/or EaD (e.g., the utilized (drawn) amount increases materially), but not the PD of the facility. Based on a PD-based trigger, as required by IFRS 9, this facility will not be “downgraded” to a lower credit quality bucket (e.g., from bucket 1 to bucket 2), even though the potential loss that could be incurred has increased, owing to the higher LGD and/or EaD. The lack of inclusion of LGD and EaD elements when setting the triggers hinders our ability to accurately capture the EL in a timely fashion.\(^7\)

Having stated the shortcoming of having triggers that are solely based on PD, below we outline an approach to define these PD-based triggers based on the one-year conditional PD.\(^8\)

1. The trigger to move bucket 1 corporate obligors that originated as investment grade to bucket 2: when the expected PD under the CFCE exceeds the investment grade PD threshold, the loan can no longer be considered as investment grade and must be moved from bucket 1 to bucket 2, where lifetime EL applies. That is, when

\[ E[PD \mid \text{under CFCE}] > \text{investment grade PD level threshold}. \]

The investment grade PD threshold is therefore the maximum conditional (PIT) PD allowed for an investment grade obligor. Note that some obligors will remain

\(^7\) Note that the increase in LGD and/or EaD is only captured when and if EL is recalculated.

\(^8\) Alternatively, the triggers can be set based on lifetime (cumulative) PD. This, however, creates undesirable complexity in practice for two reasons. First, the estimation of forward PD requires the capability of reliably forecasting the term structure of CFCE, which is difficult. Second, since lifetime PD is a function of the remaining time to maturity of the asset, any change in the lifetime PD could be the results of either a change in the credit quality of the borrower or simply the (natural) shortening of the time to maturity over time, or both. If an increase in lifetime PD is indeed due to the deterioration of credit quality, it warrants moving the asset to a lower bucket. However, it would be unfair to assign the asset to a higher-risk bucket if the increase in lifetime PD is simply the result of the latter. In practice, it is difficult to disentangle these two confounding effects. Given these complexities and potential estimation errors, here, we present the triggers based on the one-year conditional PD.

\(^9\) The IFRS 9 rule only refers to “high-quality credit”, which is not necessarily “investment grade”. Therefore, in practice, each bank will need to come up with its own definition of high-quality credit and the corresponding bucketing rule. In the proposed bucketing rule, we are defining high-quality credit as investment-grade obligors.
in bucket 1 despite the material increase in their PD, as long as they remain at investment grade. This can be justified from the standpoint that, despite the increase in EL due to an increase in PD, the absolute value of the PD remains low enough (as investment grade) not to warrant a migration to bucket 2.

This can work in reverse: the obligors move from bucket 2 back to bucket 1 when the credit quality improves, satisfying

\[ E[PD \mid \text{under CFCE}] \leq \text{investment grade PD level threshold}. \]

(2) The trigger to move bucket 1 corporate obligors that originated as noninvestment grade to bucket 2: as these obligors originated as noninvestment grade, their current PDs are likely to be higher than the investment grade conditional PD threshold mentioned above. To define the trigger for moving from bucket 1 to 2, we therefore instead use a proportional threshold that is a benchmark against which to test credit quality at origination:

\[
\frac{E[PD \mid \text{CFCE}] - E[PD \mid \text{FCE at origination}]}{E[PD \mid \text{FCE at origination}]} > \% \text{ threshold},
\]

where \( E[PD \mid \text{FCE at origination}] \) is the expected PD given the forecast of the credit environment at the origination of the loan (\( \text{FCE at origination} \)). The proportional threshold (\% threshold) is chosen to reflect a material enough relative increase in the conditional PD (eg, 10\%). This trigger can also work in reverse, and the obligor moves from bucket 2 back to bucket 1 when its credit quality improves, satisfying

\[
\frac{E[PD \mid \text{CFCE}] - E[PD \mid \text{FCE at origination}]}{E[PD \mid \text{FCE at origination}]} \leq \% \text{ threshold}.
\]

(3) The trigger to move retail exposures from bucket 1 to bucket 2: typically retail obligors are pooled, and therefore are not classified into investment versus noninvestment grades. The above triggers for corporate obligors are therefore not applicable to retail exposures. We should nevertheless follow the same principle. In order to ensure the movement between buckets will only be triggered by a material change in EL, the thresholds need to be dependent on the PD level. That is, if the PD is sufficiently low to begin with, its increasing by multiples should not necessarily warrant moving the asset from bucket 1 to 2, as the absolute level of PD is still considered to be low after the increase. For example, an increase from a PD of 0.15\% to 0.45\% – a threefold increase – may not be enough to warrant a downgrade; whereas an increase from a PD of 5.00\% to
7.50% – only a 50% increase – may do so. This could be accommodated by imposing a double-trigger requirement:

\[ E[PD \mid \text{under CFCE}] > \text{PD level threshold} \]

and

\[ \frac{E[PD \mid \text{CFCE}] - E[PD \mid \text{FCE}^\text{at origination}]}{E[PD \mid \text{FCE}^\text{at origination}]} > \% \text{ threshold.} \]  \hspace{1cm} (2.7)

The (double) trigger also works in reverse. The asset will be moved from bucket 2 back to bucket 1 when the credit quality improves, satisfying

\[ E[PD \mid \text{under CFCE}] \leq \text{PD level threshold} \]

or

\[ \frac{E[PD \mid \text{CFCE}] - E[PD \mid \text{FCE}^\text{at origination}]}{E[PD \mid \text{FCE}^\text{at origination}]} \leq \% \text{ threshold.} \]

(4) The trigger to move from buckets 1 and 2 to bucket 3: nonperforming obligors will be moved from buckets 1 and 2 to bucket 3 when their conditional PDs exceed the “performing” grade PD threshold, that is, when

\[ E[PD \mid \text{under CFCE}] > \text{performing grade PD threshold.} \]  \hspace{1cm} (2.8)

The performing grade PD threshold is likely to be above 50%, and needs to be defined in the institution’s internal policies.

To ensure that we are able to capture both quantitative and qualitative information in a timely fashion, the mechanical thresholds discussed above should be supplemented with expert-judgment-based thresholds devised based on a bank’s experience of its specific credit portfolios. For example, besides judging based on the conditional PD of the obligor, the bank may have other information that will suggest a loan should be considered as nonperforming (eg, the loan is already ninety days past due).

### 2.3 Procyclicality in provision requirement under IFRS 9

There are two sources of procyclicality in credit provision introduced by IFRS 9. First, the one-year EL measure conditional on CFCE is deemed to be procyclical, as all of its components, namely PD, expected LGD and expected EaD, are functions of the prevailing economic condition and thus PIT in nature. This EL measure under IFRS 9 is expected to be more procyclical than the EL based on A-IRB parameters. This is because, in practice, the PD parameter under A-IRB is likely to be less PIT than that under IFRS 9, not to mention the LGD and EaD parameters under A-IRB.
FIGURE 1 Adoption of IFRS 9 will amplify the procyclicality of credit provision.

are typically TTC. Second, the procyclicality will be amplified as obligors switch from bucket 1 to bucket 2 during a downturn, while switching from bucket 2 back to bucket 1 as the credit condition improves. Figure 1 illustrates how EL will behave over two business cycles under IFRS 9. Suppose initially all obligors are in bucket 1. When the credit condition starts to degenerate, the one-year EL increases given the higher estimations of PD and expected LGD and EaD. As the credit condition continues to degenerate, the PD-based threshold is breached for (at least some of) the obligors. These obligors are thus moved to bucket 2. The switching from a one-year EL to a lifetime EL measure by itself will result in a higher overall EL, thus amplifying the effect of the worsening market condition. As the credit condition finally starts to improve and eventually (perhaps some of) the bucket 2 obligors’ PD becomes lower than the respective thresholds, they are moved from bucket 2 back to bucket 1. Switching from a lifetime EL back to a one-year EL measure will result in a significant drop in the overall EL, again amplifying the sensitivity of EL to the changing business cycle conditions. It is important to note that the amplification as a result of switching between buckets is directly proportional to the duration of the portfolio. The longer the duration, the greater the difference between one-year EL and lifetime EL, and the more procyclicality there will be.

A natural extension to this discussion is the increased disconnect between the capital requirement (under Basel II) and the credit provisions (under IFRS 9). Note that the capital requirement is defined net of the one-year EL under both A-IRB’s regulatory and economic capital frameworks. The fact that IFRS 9’s one-year EL is more procyclical than A-IRB’s (always one-year) EL introduces the first disconnect between the two measures. The more significant issue, however, is the switching between one-year EL and lifetime EL under IFRS 9. When measuring the provision based on
lifetime EL while the capital requirement is net of the one-year EL, we are indeed double counting the credit loss by the difference between the one-year EL and lifetime EL. This is an issue arising from the disconnect between the risk horizons assumed in the capital requirement and reserve estimations. While the former is always one year, the latter may switch to lifetime. It may be argued that, although the prescribed risk horizon is one year under the Basel II Pillar 1 regulatory capital requirement, banks do assess their capital adequacy over a longer horizon in conducting their ICAAP and/or CCAR. We understand at the time of writing that when IFRS 9-based reserves exceed the one-year EL under A-IRB, banks will be able to recognize some Tier 1 capital benefit up to a certain threshold. Nevertheless, any remaining double counting or disconnect between the capital requirement and reserves requirement will add further complexity and potential surprises to the banks’ management of their reserve and capital levels.

3 ESTIMATION OF PROBABILITY OF DEFAULT

In this section, we examine how we may adapt the stress testing models commonly used in fulfilling Basel II A-IRB requirements for calculating conditional PD under IFRS 9. The so-called top-down PD stress testing models exploit the statistical relation between systematic PD implicit in a credit portfolio and macroeconomic variables that govern the underlying credit risk. For example, building on the single-factor infinitely granular portfolio credit risk model of Vasicek (1987), Miu and Ozdemir (2009) proposed a time series model to calculate risk-rating-specific PD under predefined stressed scenarios that could be articulated with observable macroeconomic variables. Besides being used to generate stressed PD, the model can also be used to calculate PD conditional on the current or expected outlook of the economy. Thus, it not only serves as a tool for assessing risk capital requirement under A-IRB, but can also be used to calculate forward-looking conditional PD in satisfying IFRS 9. In the following, we outline the conditional PD model adapted from Miu and Ozdemir (2009) and demonstrate the implementation of such a model in assessing the conditional PD for a representative residential mortgage portfolio.

Suppose borrowers are uniform in terms of their credit risks within a certain segment (e.g., a specific risk rating) of the portfolio. Their individual PD risk $p_i^t$ at time $t$ is driven by both the systematic PD risk $P_t$ and the borrower-specific PD risk $e_i^t$.\footnote{This approach of decomposing an obligor’s default risk into its systematic and idiosyncratic components is consistent with the model underlying the Basel II, Pillar I risk-weight function.}

For example, for borrower $i$, \[ p_i^t = R \times P_t + \sqrt{1 - R^2} \times e_i^t. \] (3.1)
Both $P_t$ and $e_{PD,t}$ are assumed to follow the standard normal distribution. Thus, $p_t$ also follows the standard normal distribution. Under Merton’s framework (Merton 1973), we can interpret $p_t$ as a latent variable, which is a normalized function of the borrower’s asset return. For retail credit facilities, we may interpret $p_t$ as a normalized measure of the financial health of the individual borrower, which varies with both systematic factors and borrower-specific conditions. The borrower defaults on their loan when $p_t$ becomes less than some constant default point (DP). Thus, the smaller the value of $p_t$ (i.e., the closer to DP), the greater the borrower’s PD. The coefficient $R$ is assumed to be uniform across borrowers and measures the sensitivity of individual risks to the systematic PD risk, $P_t$. The parameter $R^2$ is therefore the pairwise correlation of $p_t$ between borrowers as a result of the systematic risk factor. Equation (3.1) is in fact the single-factor model considered by Vasicek (1987) in deriving the loss distribution of a credit portfolio.\(^{11}\)

We can define the long-run probability of default (LRPD) of a particular risk rating $m$ (where $m = 1, 2, \ldots, M$) as the unconditional probability of $p_t$ being lower than the risk-rating-specific default point (DP\(_m\)):\(^{12}\)

$$\text{LRPD}_m = \Pr[p_t^i < \text{DP}_m].$$

LRPD is therefore a function of DP\(_m\). This function is defined by the unconditional distribution of $p_t$, which is in turn governed by the unconditional distribution of $P_t$ via (3.1). To allow for the computation of conditional PD, we can model the systematic PD risk $P_t$ as a function $f(\cdot)$ of, say, $J$ explanatory variables $X_t^1, X_t^2, \ldots, X_t^J$

$$P_t = f(X_t^1, X_t^2, \ldots, X_t^J, X_{t-1}^1, X_{t-1}^2, \ldots, X_{t-1}^J) + \varepsilon.$$

These explanatory variables could be macroeconomic variables, market variables and/or economic indicators that are expected to be able to explain the systematic PD risk

---

\(^{11}\) The single-factor Vasicek model under Merton’s framework is arguably more relevant for modeling the credit risk of wholesale portfolios than that of retail portfolios, given that the notion of “asset return” is more appropriate for obligors in the former than in the latter. Nevertheless, it is a commonly adopted approach in modeling the PD of both kinds of portfolios. For example, the calculations of Basel II, Pillar 1 risk-weighted assets for both wholesale and retail portfolios are formulated based on the single-factor Vasicek model. It is also important to emphasize that we are not assuming different kinds of retail portfolios (e.g., mortgages, credit cards, etc) are driven by the same single risk factor. Separate conditional PD models are constructed for different kinds of retail portfolios, each driven by their own specific risk factor. Finally, the pairwise correlation $R^2$ is also portfolio specific. For example, wholesale portfolios tend to have a higher pairwise correlation than retail portfolios. In the calibration of long-run probability of default (LRPD) and the generation of conditional PD, we need to estimate and use the appropriate pairwise correlation for the specific portfolio under consideration.

\(^{12}\) Note that superscript $i$ denotes the $i$th borrower within the uniform risk rating.
of the credit portfolio under consideration. Some examples of explanatory variables are: real GDP growth rate; unemployment rate; house price index; interest rate; stock market index return. Note that, in (3.3), besides the contemporaneous values of the explanatory variables, we may also include lagged values of these variables as possible additional explanatory variables. The first term of (3.3) (ie, $f(\cdot)$) may be interpreted as the explainable component of $P_t$, while the second term (ie, the residual term $e_t$) is the unexplainable component. Under this framework LRPD, $R^2$ and the other parameters governing function $f(\cdot)$ can be estimated by observing the time series of historical default rates and explanatory variables (for details, see Miu and Ozdemir 2009).13

After calibrating the model, we can then use it to generate risk-rating-specific PD conditional on the current and expected economic outlooks so as to satisfy IFRS 9. Specifically, the term structure of the forward one-year conditional PD over the first, second and third years (PD$_1$, PD$_2$ and PD$_3$, respectively) can be calculated by evaluating the following conditional probabilities according to the term profile of the expected values $x$ of the explanatory variables $X$. For example, for risk rating $m$,

$$PD_{m,1} = Pr[p_1^i < DP_m | X_1^1 = x_1^1, X_2^1 = x_2^1, \ldots, X_J^1 = x_J^1], \quad (3.4a)$$

$$PD_{m,2} = Pr[p_2^i < DP_m | X_2^2 = x_2^2, X_2^2 = x_2^2, \ldots, X_J^2 = x_J^2], \quad (3.4b)$$

$$PD_{m,3} = Pr[p_3^i < DP_m | X_3^3 = x_3^3, X_3^3 = x_3^3, \ldots, X_J^3 = x_J^3]. \quad (3.4c)$$

We now provide an example of implementing the above conditional PD model on a representative residential mortgage portfolio. Similar to many other credit portfolios, we do not have a sufficiently long historical default rate data series for this mortgage portfolio to calibrate the above model robustly. An external proxy is therefore used in the selection of explanatory variables and the calibration of the parameters of the function $f(\cdot)$.14 We use the publicly available proportion of mortgages in arrears at the national level as our default rate proxy. We thus assume the same set of explanatory variables is driving both the proportion of mortgages in arrears and the default rate of our mortgage portfolio. The model development involves the following two-step process.

---

13 By formulating the systematic PD risk $P_t$ as a function of multiple explanatory variables in (3.3), we are in effect extending the single-factor representation of default risk to a multifactor one. This multifactor representation allows for more flexibility in modeling the conditional PD for different kinds of credit portfolios.

14 The use of an external proxy may be challenged by model validation on data representativeness and key assumption risk. Therefore, it is important that representativeness of the external proxy is justified and a robust methodology is employed to explicitly account for the differences in asset correlations and the level of PDs between the internal portfolio and the external proxy (see Miu and Ozdemir (2008) for an example of such a methodology).
TABLE 1  Estimated coefficients of the best performing model.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter</th>
<th>Point estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>$a$</td>
<td>$-17.441$</td>
</tr>
<tr>
<td>Annual growth rate of real GDP</td>
<td>$b_1$</td>
<td>$0.113$</td>
</tr>
<tr>
<td>Annual change of unemployment rate</td>
<td>$b_2$</td>
<td>$-0.150$</td>
</tr>
<tr>
<td>Quarterly growth rate of residential property resale price</td>
<td>$b_3$</td>
<td>$0.050$</td>
</tr>
<tr>
<td>Detrended value of number of housing units started</td>
<td>$b_4$</td>
<td>$0.111$</td>
</tr>
<tr>
<td>Quarterly change in five-year residential mortgage rate</td>
<td>$b_5$</td>
<td>$0.022$</td>
</tr>
</tbody>
</table>

Step 1: using the historical data of our default risk proxy, we compare the explanatory power of different specifications of $f(\cdot)$ by considering different combinations of potential explanatory variables. Essentially, we assume the systematic PD risk, $P_t$, implicit in the external proxy is identical to that of our mortgage portfolio. We conduct our estimations using both single-variable and multiple-variable specifications of $f(\cdot)$ in order to identify the specification that is both intuitive and of the highest explanatory power.

Step 2: we estimate the default point, $DP_m$, using the (limited) internal default rate data of our mortgage portfolio. Here, we assume the function $f(\cdot)$ of the “best-fitted” model identified and calibrated in step 1 is equally applicable to our mortgage portfolio. An appropriate value of $R^2$ is also selected so that the conditional PD outputs of the calibrated model match the observed default rates of our mortgage portfolio as close as possible.

A total of twenty-eight variables are considered in step 1 in testing their explanatory power on the systematic PD risk implicit in the proxy. They include general economic variables (eg, real GDP, unemployment rate), various interest rates, exchange rate, stock market index, confidence index and various housing market data, and are considered to be key drivers/indicators of mortgage credit risks.

We consider a sample period from 1986 to 2014. For each of these variables, we consider different versions of the time series, including its level, annual change, quarterly change, annual proportional change, quarterly proportional change and its detrended value.$^{15}$

We specify a linear representation of the function $f(\cdot)$:

$$f(\cdot) = a + b_1 X_{t1} + \cdots + b_J X_{tJ}$$  \hspace{1cm} (3.5)

$^{15}$ Standard stationary tests are conducted on the variables to ensure nonstationary time series will be excluded from the modeling exercise.
and measure model performance based on in-sample goodness-of-fit over our sample period. The best performing model is made up of five explanatory variables: annual growth rate of real GDP; annual change of unemployment rate; quarterly growth rate of residential property resale price; detrended value of the number of housing units started; and quarterly change in five-year residential mortgage rate. The estimated coefficient values are reported in Table 1.

After calibrating the time series model, we then calculate the DP and LRPD for each of the risk ratings of the mortgage portfolio. Finally, the one-year PD of each risk rating can be calculated conditional on the realization of the five underlying explanatory variables at different points in time. In Figure 2, we plot the conditional one-year PD of the aggregated mortgage portfolio (ie, across all risk ratings) based on actual values of the five variables realized from 1991 to 2014. For comparison, we also plot the realized default rates at the portfolio level from 2010 to 2013.

4 ESTIMATION OF LOSS GIVEN DEFAULT

In this section, we present a methodology to predict the “term structure” of LGD of secured facilities by modeling the “term structure” of the value of its underlying collateral. Collateral value is one of the most significant drivers of the ultimate recovery value (and thus the LGD) of a defaulted secured instrument. For example, the recovery value from a defaulted residential mortgage should be closely related to the value of the foreclosed housing property that will be disposed of during the workout

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process. The same can be said for a defaulted equipment financing contract, of which the recovery value is governed by the residual value (net of any depreciation as a result of wear and tear) of the equipment itself. Thus, if we can accurately forecast the value of the collateral, we will also be able to project the term structure of the LGD of the secured facility for the calculation of EL under IFRS 9.16

Before we introduce our model, let us start by stating some basic definitions and notation. Suppose $T_{\text{default}}$ denotes the time of default of a secured facility. The percentage and dollar amount of LGD can be defined as

$$\text{LGD}_{T_{\text{default}}} \% = \left(1 - \frac{\text{PV}_{T_{\text{default}}}(\text{net recoveries})}{\text{EaD}_{T_{\text{default}}}}\right)$$

(4.1)

and

$$\text{LGD}_{T_{\text{default}}} \$ = \text{LGD}_{T_{\text{default}}} \% \times \text{EaD}_{T_{\text{default}}},$$

(4.2)

where $\text{EaD}_{T_{\text{default}}}$ is the EaD of the facility as measured at default and the present value of recovery is the sum of net cashflows received during the workout process capitalized at an appropriate discount rate $r$. That is,

$$\text{PV}_{T_{\text{default}}}(\text{net recoveries}) = \sum_{m=1}^{M} (\text{recovery}_{t_m} - \text{cost}_{t_m}) \times e^{-r \times (t_m - T_{\text{default}})},$$

(4.3)

where there are $M$ cashflows occurring at time $t_1, t_2, \ldots, t_M$ with $T_{\text{default}} \leq t_1 \leq t_2 \leq \cdots \leq t_M$.

The present value of the recovery value is expected to be quite similar to the value of the underlying collateral measured at the time of default ($V_{T_{\text{default}}}$). Their difference is mainly attributable to

- the costs incurred in the recovery process (eg, legal fees, collection fees),
- the errors in the estimation of the collateral value at the time of default (eg, the appraisal value of a residential property at the time of default may be biased, the amortized value of leased equipment after depreciation may differ from its actual economic value).

Both the costs of recovery and the errors in collateral valuation are expected to be relatively insensitive to the prevailing state of the business cycle. One of the main assumptions of the proposed model is that the present value of recovery is assumed to be a fixed fraction ($\delta$) of the value of the underlying collateral at the time of default ($V_{T_{\text{default}}}$). For a facility collateralized on a single asset, we therefore have

$$\text{PV}_{T_{\text{default}}}(\text{net recoveries}) = \delta \times V_{T_{\text{default}}},$$

(4.4)

16 To forecast the LGD of unsecured facilities, one may wish to refer to the conditional LGD models examined by Ozdemir and Miu (2008).
We can refer to $\delta$ as the “net recovery ratio”, representing the proportion of collateral value at default that is recouped from the write-off/recovery process. We assume the value of $\delta$ is uniform across the same type of secured facilities (e.g., a certain segment of residential mortgages) and thus can be estimated by taking the average of the ratios $PV_{T_{\text{default}}} \left( \text{net recoveries} \right) / V_{T_{\text{default}}}$ for all defaulted loans within that specific secured facility type over a certain historical sample period. By specifying the recovery value as a fixed fraction of the collateral value, we can then predict the recovery value (and thus the LGD) by explicitly modeling the variation of the asset values underlying the collateral as functions of observable macroeconomic factors (e.g., house price index, equipment price index).\(^{17}\) Note that we can readily extend the above specification to cover those facilities against which multiple collaterals are pledged. For example, the recovery value of a loan collateralized on two different assets can be expressed as

$$PV_{T_{\text{default}}} \left( \text{net recoveries} \right) = \delta_1 \times V_{1,T_{\text{default}}} + \delta_2 \times V_{2,T_{\text{default}}}, \quad (4.5)$$

where $V_{1,T_{\text{default}}}$ and $V_{2,T_{\text{default}}}$ are the collateral values of the two assets at the time of default, and $\delta_1$ and $\delta_2$ are the respective net recovery ratios applicable to the two kinds of assets.

In order to predict the recovery value conditional on the current and/or expected economic outlooks, we propose the following regression model for the growth rate of the collateral value. Suppose today is time $t_0$ and we want to predict LGD for a default that will occur at future time $t_0 + \tau$. Let $r_{t_0,t_0+\tau}^V$ be the annualized growth rate of the collateral value from time $t_0$ to $t_0 + \tau$. Thus,

$$V_{t_0+\tau} = V_{t_0} \times \exp(\tau r_{t_0,t_0+\tau}^V). \quad (4.6)$$

The expected economic outlook is supposed to dictate the expected changes in collateral value and, in turn, its growth rate ($r_{t_0,t_0+\tau}^V$). For example, if we think the general residential property price (as measured by a certain house price index) will increase by 5% in the next twelve months, it is not unreasonable to expect that the collateral value of the residential property underlying a mortgage will tend to increase at a similar rate. On the other hand, depreciation (e.g., in the form of wear and tear for leased equipment) will tend to result in a “negative drag” on the growth rate of collateral value. Finally, the appraised value of the collateral could be subject to different kinds of errors and/or biases when the loan is still performing versus when it has defaulted. Since, in applying (4.6), we are in fact predicting $V_{t_0+\tau}$ and interpreting it as $V_{T_{\text{default}}}$ (i.e., $T_{\text{default}} = t_0 + \tau$) when using it in (4.4) to calculate the recovery value for a default that will occur at time $t_0 + \tau$, we also need to incorporate such differences

\(^{17}\)While the assumption of recovery value as a fixed fraction of the collateral value is considered to be a practical solution, it should be stated that the ratio of recovery value to collateral value is likely to be sensitive to market supply and demand, which may vary with macroeconomic conditions.
in appraisal errors/biases in the change in collateral values. To incorporate the above
determinants of the growth rate of the collateral value, we specify the following linear
representation for \( r_{V,t_0}^{t_0+t} \):

\[
    r_{V,t_0}^{t_0+t} = \alpha + \beta_1 r_{\text{factor}_1,t_0}^{t_0+t} + \beta_2 r_{\text{factor}_2,t_0}^{t_0+t} + \cdots + \beta_J r_{\text{factor}_J,t_0}^{t_0+t} + \epsilon_{t_0},
\]

(4.7)

where \( r_{\text{factor}_1,t_0}^{t_0+t}, r_{\text{factor}_2,t_0}^{t_0+t}, \ldots, r_{\text{factor}_J,t_0}^{t_0+t} \) are the annualized rates of changes of possibly
\( J \) underlying drivers of the collateral value (eg, for residential mortgages, these fac-
tors may simply be different residential property resale price indexes; for equipment
leasing, these factors may be different new and/or resale equipment price indexes).
The coefficients \( \beta_1, \beta_2, \ldots, \beta_J \) measure the sensitivities of the collateral value of the
specific asset under consideration to the different factors/indexes. Any depreciation
will be represented by a negative intercept \( \alpha \) in (4.7). The same intercept will also
capture any bias in the appraised value that is independent of any of the factors. If
the bias in the appraised value is somehow a function of the factors (eg, the appraised
value tends to be too optimistic (pessimistic) when the market condition tends to be
good (bad)), it will also show up in the coefficients \( \beta_1, \beta_2, \ldots, \beta_J \). In practice, we
will be working with the expected version of (4.7) and thus ignoring the random errors
\( \epsilon_{t_0} \):

\[
    E_{t_0} [r_{V,t_0}^{t_0+t}] = \alpha + \beta_1 E_{t_0} [r_{\text{factor}_1,t_0}^{t_0+t}] + \beta_2 E_{t_0} [r_{\text{factor}_2,t_0}^{t_0+t}] + \cdots + \beta_J E_{t_0} [r_{\text{factor}_J,t_0}^{t_0+t}].
\]

(4.8)

Thus, according to (4.6), our prediction of the collateral value at the time of default,
\( T_{\text{default}} = t_0 + \tau \), is

\[
    E_{t_0} [V_{T_{\text{default}}}] = E_{t_0} [V_{t_0+t}] = V_{t_0} \times \exp(\tau E_{t_0} [r_{V,t_0}^{t_0+t}]).
\]

(4.9)

As stated, this approach for collateral value growth rate is particularly suitable for
mortgages or secured lending with specific collaterals whereby there is a tangible way
of estimating the collateral value. This approach would be less suitable for blanket lien
types of collateral. Then, according to (4.4), the expected recovery value at default
can be expressed as

\[
    E_{t_0} [PV_{T_{\text{default}}}=t_0+t] = \delta \times V_{t_0} \times \exp(\tau E_{t_0} [r_{V,t_0}^{t_0+t}]).
\]

(4.10)

\[18\] Without loss of generality, in specifying (4.7), we are assuming the intercept \( \alpha \), which captures
the rate of depreciation and the bias in appraised value, is a constant with respect to the prediction
horizon \( \tau \). We can easily generalize the model by specifying \( \alpha \) to be a function of \( \tau \). This will cater
for the situation in which the rate of depreciation is expected to be changing over the term of the
contract, according to prior experience with that particular kind of asset.
Finally, according to (4.1), the conditional expected LGD for a default time that is \( \tau \) periods from today is

\[
E_{t_0}[\text{LGD}_{T_{\text{default}}=t_0+\tau\%}] = 
\left(1 - \frac{\delta V_{t_0} \exp(\tau E_{t_0} [r_{t_0,t_0+\tau}])}{E_{t_0}[\text{EaD}_{T_{\text{default}}=t_0+\tau}]}\right) 
= 
\left(1 - \frac{\delta V_{t_0} \exp(\tau \{\alpha + \beta_1 E_{t_0} [r_{t_0,t_0+\tau}^{\text{factor}_1}] + \beta_2 E_{t_0} [r_{t_0,t_0+\tau}^{\text{factor}_2}] + \cdots + \beta_J E_{t_0} [r_{t_0,t_0+\tau}^{\text{factor}_J}]\})}{E_{t_0}[\text{EaD}_{T_{\text{default}}=t_0+\tau}]}\right),
\tag{4.11}
\]

where \( E_{t_0}[\text{EaD}_{T_{\text{default}}=t_0+\tau}] \) is the expected value of the EaD of the facility at time \( t_0 + \tau \), which we discuss in detail in Section 5. By substituting \( \tau = 1 \) year, 2 years, \ldots, \( N \) years into (4.11), we can then generate the forward-looking term structure of the LGD. Note that, besides knowing today’s collateral value \( (V_{t_0}) \) and the expected EaD \( (E_{t_0}[\text{EaD}_{T_{\text{default}}=t_0+\tau}]) \) of the facility, to evaluate (4.11) we also need to specify the expected changes in values of the underlying factors \( (\text{ie}, E_{t_0} [r_{t_0,t_0+\tau}^{\text{factor}_1}], E_{t_0} [r_{t_0,t_0+\tau}^{\text{factor}_2}], \ldots, E_{t_0} [r_{t_0,t_0+\tau}^{\text{factor}_J}]) \) that are consistent with the current market outlook under CFCE. More importantly, we need to find out the values of the model parameters \( \delta, \alpha \) and \( \beta_1, \beta_2, \ldots, \beta_J \). As mentioned above, the net recovery ratio, \( \delta \), is assumed to be a constant for the same types of secured facilities and thus can be estimated by taking the average of the ratios \( \text{PV}_{T_{\text{default}}}(\text{net recoveries})/V_{T_{\text{default}}} \) for all defaulted loans within that specific secured facility type over a certain historical sample period. The other parameters, \( \alpha \) and \( \beta_1, \beta_2, \ldots, \beta_J \), are also supposed to be facility-type specific and can be estimated by conducting a regression analysis based on (4.7). Specifically, using historical collateral value data and the specified factors, we will regress the changes in collateral value for a certain type of facility (eg, within a segment of residential mortgages) against the contemporaneous changes in the factors of interest over a certain historical sample period. Rather than this being a purely empirical exercise, we may also want to impose some structure if some of the components that dictate the changes in collateral value are expected to behave in a deterministic fashion. For example, in equipment financing/leasing, where the equipment itself is the collateral, the equipment’s rate of amortization implicit in its prespecified amortization schedule can serve as a reference in determining the value of the intercept term, \( \alpha \).

Let us end this section with a simple numerical example. Suppose we want to calculate the LGD to be applied to a default occurring exactly one year from today. Today’s collateral value of a secured facility is $100. Suppose there is only a single underlying factor driving the collateral value of facilities belonging to this specific asset class and the estimated values of \( \delta, \alpha \) and \( \beta \) for this asset class are 0.90, –0.30
TABLE 2  Conditional LGDs under different factor value scenarios.

<table>
<thead>
<tr>
<th>( E_{t_0}[V_{t_0+1}] ) ($)</th>
<th>( E_{t_0}[\text{LGD}_{t_0+1}] ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>68.0</td>
<td>18.3</td>
</tr>
<tr>
<td>74.1</td>
<td>11.1</td>
</tr>
<tr>
<td>80.7</td>
<td>3.2</td>
</tr>
</tbody>
</table>

and 0.85, respectively. Based on an expected EaD of $75 at year end, the conditional LGDs under different expected changes in the factor value (ie, \( E_{t_0}[r_{t_0,t_0+1}^{\text{factor}}] \)) are reported in Table 2.

5 ESTIMATION OF EXPOSURE AT DEFAULT AND EXPECTED LOSS

After calculating the conditional PD and the expected LGD, the last step is to estimate EL by incorporating information regarding the expected EaD of the facility. As discussed in Section 2, if we take the usual approximation by assuming LGD and EaD are independent, we have a one-year EL given by

\[
\text{EL}_{\text{1-year}} = E_{t_0}[\text{LGD}_{t_0+1}] \times E_{t_0}[\text{EaD}_{t_0+1}] \times \text{PD}_1, \tag{5.1}
\]

where \( E_{t_0}[\text{LGD}_{t_0+1}] \) and \( E_{t_0}[\text{EaD}_{t_0+1}] \) are the expected LGD and EaD of the facility conditional on CFCE if it defaults in the first year, and \( \text{PD}_1 \) is the probability of default of the obligor in the first year conditional on CFCE. We will be calculating this one-year EL for the exposures in bucket 1. For those in bucket 2, a lifetime EL is called for. The lifetime EL of a facility that lasts for \( n \) periods can be estimated by

\[
\text{EL}_{\text{lifetime}} = E_{t_0}[\text{LGD}_{t_0+1}] \times E_{t_0}[\text{EaD}_{t_0+1}] \times \text{PD}_1 + \sum_{\tau=2}^{n} E_{t_0}[\text{LGD}_{t_0+\tau}] \times E_{t_0}[\text{EaD}_{t_0+\tau}] \times \text{PD}_{\tau} \prod_{j=1}^{\tau-1} (1 - \text{PD}_j), \tag{5.2}
\]

where \( E_{t_0}[\text{LGD}_{t_0+\tau}] \) and \( E_{t_0}[\text{EaD}_{t_0+\tau}] \) are the expected LGD and EaD of the facility conditional on CFCE if it defaults in the \( \tau \)th period from today, ie, \( t_0 \), and \( \text{PD}_{\tau} \) is the one-period conditional probability of default of the obligor over the \( \tau \)th period that is consistent with CFCE. Note that the specification in (5.2) is flexible enough to cater to a lifetime EL calculation under a discrete risk assessment frequency that is other than one year. If LGD and EaD are not expected to be constant over time, the higher the frequency, the more accurate the EL estimation will be. However, in practice, it is quite unlikely that it will be more frequent than quarterly.

With the conditional PD and LGD models presented in Sections 3 and 4, we can estimate the one-year and lifetime EL based on (5.1) and (5.2) once we are able
TABLE 3 Conditional term structure of expected loss given default and exposure at default.

(a) Conditional term structure of expected LGD and EaD

<table>
<thead>
<tr>
<th>Year</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{t_0}[r_{t_0,t_0+\tau}]$ (%)</td>
<td>-10.0</td>
<td>-10.0</td>
<td>-5.0</td>
</tr>
<tr>
<td>$E_{t_0}[V_{t_0+\tau}]$ ($)</td>
<td>407 177</td>
<td>368 429</td>
<td>387 319</td>
</tr>
<tr>
<td>$EaD_{t_0+\tau}$ ($)</td>
<td>390 000</td>
<td>375 000</td>
<td>350 000</td>
</tr>
<tr>
<td>$E_{t_0}[LGD_{t_0+\tau}]$ (%)</td>
<td>21.7</td>
<td>26.3</td>
<td>17.0</td>
</tr>
<tr>
<td>$E_{t_0}[LGD_{t_0+\tau}] \times EaD_{t_0+\tau}$ ($)</td>
<td>84 617</td>
<td>98 678</td>
<td>59 511</td>
</tr>
</tbody>
</table>

(b) Conditional term structure of expected LGD and EaD incorporating expected prepayment

<table>
<thead>
<tr>
<th>Year</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{t_0}[r_{t_0,t_0+\tau}]$ (%)</td>
<td>-10.0</td>
<td>-10.0</td>
<td>-5.0</td>
</tr>
<tr>
<td>$E_{t_0}[V_{t_0+\tau}]$ ($)</td>
<td>407 177</td>
<td>368 429</td>
<td>387 319</td>
</tr>
<tr>
<td>$E_{t_0}[LGD_{t_0+\tau}]$ (%)</td>
<td>21.7</td>
<td>26.3</td>
<td>17.0</td>
</tr>
<tr>
<td>$EaD_{t_0+\tau}$ ($)</td>
<td>390 000</td>
<td>375 000</td>
<td>350 000</td>
</tr>
<tr>
<td>$E_{t_0}[\theta_{t_0+\tau}]$ (%)</td>
<td>7.0</td>
<td>10.0</td>
<td>14.0</td>
</tr>
<tr>
<td>$E_{t_0}[EaD_{t_0+\tau}]$ ($)</td>
<td>362 700</td>
<td>337 500</td>
<td>301 000</td>
</tr>
<tr>
<td>$E_{t_0}[LGD_{t_0+\tau}] \times E_{t_0}[EaD_{t_0+\tau}]$ ($)</td>
<td>78 706</td>
<td>88 763</td>
<td>51 170</td>
</tr>
</tbody>
</table>

to determine the expected EaD profile of our facility. The calculation will be quite straightforward if the EaD profile of the facility is deterministic over time. However, if the EaD profile is expected to be stochastic, we need to conduct further estimations for the credit conversion factors, as illustrated below.

5.1 Deterministic EaD

For lots of secured facilities (e.g., residential mortgages, term loans, amortizing loans), EaD follows a deterministic path as dictated by the outstanding balance over the life of the loan. The EaD profile is known and prespecified at initiation (ignoring any prepayments or payments missed prior to potential default). There is, therefore, no uncertainty regarding the EaD value throughout the life of the facility, and thus $E_{t_0}[EaD_{t_0+\tau}]$ becomes $EaD_{t_0+\tau}$ when we evaluate (5.1) and (5.2).

Let us illustrate the one-year and lifetime EL estimation with a numerical example. Suppose the outstanding balance of a residential mortgage is currently $400 000. The remaining term is three years. According to the amortization schedule, the outstanding
balances at the end of the next three years are $390,000, $375,000 and $350,000, respectively. Today’s appraised value \((V_t)\) of the residential property is $450,000. Suppose there is only a single underlying factor (e.g., a national house price index) driving the value of the residential property. The estimated LGD parameters \(\delta, \alpha\) and \(\beta\) for this asset class are 0.75, 0.00 and 1.00, respectively. Suppose the annualized changes in the underlying house price index are expected to be \(-10\%\), \(-10\%\) and \(-5\%\) over the next year, the second next year and the third year, respectively. In Table 3(a), we report the predicted collateral values \((E_t[V_{t+\tau}])\) at the end of the next three years using (4.9) and based on the above expected changes in the underlying house price index. Then, using (4.11), we calculate the expected LGD profile \((E_t[LGD_{t+\tau}])\) of this mortgage over the next three years, which is also presented in Table 3(a).

In the last row of Table 3(a), we report the expected loss (in dollars) of the mortgage contingent on the default occurring in each of the next three years. Further, suppose the one-year conditional PD of the mortgagor is 5% for each of the next three years. The one-year EL of this mortgage is therefore equal to $84,617 \times 5\% = $4231, whereas its lifetime EL is

\[
84,617 \times 5\% + 98,678 \times 5\% \times 95\% + 59,511 \times 5\% \times 95\% \times 95\% = 11,604.
\]

### 5.1.1 Incorporating prepayments

For certain products, the borrower is allowed to make prepayments up to a certain threshold. In this case, we can use a more general case for \(E_t[EaD_{t+\tau}]\) when we evaluate (5.1) and (5.2), that is,

\[
E_t[EaD_{t+\tau}] = (1 - E_t[\theta_{t+\tau}]) \times EaD_{t+\tau},
\]

where \(E_t[\theta_{t+\tau}]\) is the expected prepayment rate that can be estimated from historical data. In Table 3(b), we present the results of the previous numerical example while incorporating expected prepayment rate.

With the assumed prepayment rate, the one-year EL of the mortgage becomes equal to $78,706 \times 5\% = $3935; whereas its lifetime EL is

\[
78,706 \times 5\% + 88,763 \times 5\% \times 95\% + 51,170 \times 5\% \times 95\% \times 95\% = 10,461.
\]

### 5.2 Stochastic EaD

Many facilities have a stochastic EaD profile over time. For example, most of the time, the utilized amount \((U)\) of a line of credit is only a fraction of its authorized amount \((A)\). The decision to increase or decrease the drawn amount is largely at the discretion of the borrower. It is expected that the decision process is not completely

---

19 Exceeding these preestablished thresholds may be subject to penalties.
random. For example, a generally bad business condition may induce more borrowers to draw down on their credit lines, whereas it should not be too surprising that the drawn amount tends to be lower in a booming market condition. More importantly, we expect borrowers to become more aggressive in utilizing their credit lines once default is imminent. We therefore expect the use of any undrawn amount (i.e., the remaining balance of the authorized amount) to be highest during the year prior to the default event. In the Basel II A-IRB approach, in order to find out the expected EaD over the remaining life of the loan contract, it is typical to adopt a time series model based on the estimated “conversion rate” from the undrawn to the drawn amount over time, which is usually referred to as the credit conversion factor (CCF). Essentially, we assume facilities belonging to the same uniform segment (e.g., a certain kind of line of credit) behave similarly in terms of how the authorized amount is utilized and thus share the same CCF. Specifically, if the undrawn amount at the end of last period is \( A_{t-1} - U_{t-1} \), the additional amount utilized (\( \Delta U_t \)) in this period is assumed to be given by

\[
\Delta U_t = (U_t - U_{t-1}) = CCF_t \times (A_{t-1} - U_{t-1}),
\]

i.e.,

\[
U_t = U_{t-1} + CCF_t \times (A_{t-1} - U_{t-1}).
\]

Note that a subscript \( t \) is added to CCF to represent the most general case where CCF may vary over time according to the state of the business cycle. Note that the utilized amount at the end of period \( t \) (\( U_t \)) will become the EaD if the borrower defaults in period \( t \). That is,

\[
EaD_t = U_{t-1} + CCF_{ND} \times (A_{t-1} - U_{t-1}).
\]

In Basel II A-IRB, the risk horizon is one year, and therefore \( t \) is measured on a yearly basis. That is, EaD is defined over a one-year period.

As mentioned previously, we expect the conversion rate in the default year to be quite different from that in other years. We therefore use superscripts “D” and “ND” to differentiate the CCF in default year and nondefault year, respectively. Thus, we need to rewrite (5.4) and (5.5) as follows:

\[
U_t = U_{t-1} + CCF_{ND} \times (A_{t-1} - U_{t-1}),
\]

\[
EaD_t = U_{t-1} + CCF_{D} \times (A_{t-1} - U_{t-1}).
\]

Equation (5.6) therefore represents the change in utilization amount during the years in which default has not occurred, whereas (5.7) presents how EaD is formulated in the default year. In general, we expect \( CCF_D \) to be higher than \( CCF_{ND} \).

The two CCFs can be estimated from historical data by tracking the variation of the utilization rates of the facilities over time. Facilities in each uniform segment
(ie, a certain kind of facility) are assumed to share the same CCF. In A-IRB, CCF\textsuperscript{D} is typically assumed to be constant over time, ie,

\[
\text{CCF}\textsubscript{1}\textsuperscript{D} = \text{CCF}\textsubscript{2}\textsuperscript{D} = \cdots = \text{CCF}\textsubscript{n}\textsuperscript{D} = \text{CCF}\textsuperscript{D}.
\]  

(5.8)

This constant can be estimated from historical data by taking the average of the following ratios observed during the year prior to the respective default dates across all defaulted facilities of that uniform segment:

\[
\frac{\text{EaD}\textsubscript{realized} - U_{T\text{default} - 1}}{A_{T\text{default} - 1} - U_{T\text{default} - 1}},
\]

(5.9)

where EaD\textsubscript{realized} is the actual realized EaD of the defaulted facility (ie, the outstanding balance of the facility) at default time \(T_{\text{default}}\), \(U_{T_{\text{default} - 1}}\) is the utilized dollar amount at \(T_{\text{default} - 1}\) (ie, one year prior to the default date) and \(A_{T_{\text{default} - 1}}\) is the authorized dollar amount measured at the same time.

The estimation of the CCF for the nondefault year (CCF\textsuperscript{ND}) is likely to be more involved. We expect CCF\textsuperscript{ND} to vary with the business cycle. First of all, given more difficult business conditions, it is quite likely that borrowers will draw down more during bad years than in good years. On the other hand, it is also possible that, in order to take advantage of a good economy, borrowers in fact use more of their credit lines in good times than in bad times. In any case, the behavior is quite specific to the particular kind of facility. To satisfy IFRS 9, which calls for a conditional EaD assessment based on the CFCE, we need to model CCF\textsuperscript{ND} as a function of the prevailing business condition. Suppose we expect CCF\textsuperscript{ND} to be driven by the growth rate of real GDP. We can construct, say, three subsamples from our historical sample period based on the GDP growth rate realized in each year. In particular, the first subsample contains those years with a GDP growth rate within the fastest one-third of all years. In the second subsample, the GDP growth rate is within the middle third, and in the third subsample it is within the slowest third. We then calculate the average of the following ratios for all nondefaulting facilities within a uniform segment separately for each of the three subsample periods:

\[
\frac{U_t - U_{t - 1}}{A_{t - 1} - U_{t - 1}},
\]

(5.10)

where \(U_t\) and \(U_{t - 1}\) are the utilized dollar amounts of the nondefaulting facility observed at time \(t\) and \(t - 1\), respectively (ie, the two observations are exactly one year apart), and \(A_{t - 1}\) is the authorized dollar amount at time \(t - 1\). We therefore obtain three estimations of CCF\textsuperscript{ND} applicable to good, neutral and bad business conditions, respectively. In projecting the EaD profile for the EL calculation, we will therefore pick the appropriate CCF\textsuperscript{ND} based on our expectation of the prevailing real
GDP growth rate. Without the calibration of such a conditional \( \text{CCF}^{\text{ND}} \), one conservative approach is to simply assume \( \text{CCF}^{\text{ND}} \) takes the same value as the previously estimated \( \text{CCF}^{\text{D}} \). This is conservative because \( \text{CCF}^{\text{D}} \) is usually believed to be higher than \( \text{CCF}^{\text{ND}} \). Using \( \text{CCF}^{\text{D}} \) in projecting the changes in the utilized values for the nondefaulting years as well will therefore result in a larger \( \text{EaD} \).

With the estimated CCFs, we can then project the expected utilized dollar amount and \( \text{EaD} \) for each facility within the same segment over the remaining life of the contract, based on the following recursive equations. Specifically, for the years prior to the default year, the expected utilized amount is given by

\[
E[U_{t}] = E[U_{t-1}] + E[\text{CCF}^{\text{ND}}_{t}] \times (A_{t-1} - E[U_{t-1}])
= E[U_{t-1}] \times (1 - E[\text{CCF}^{\text{ND}}_{t}]) + A_{t-1} \times E[\text{CCF}^{\text{ND}}_{t}]. \tag{5.11}
\]

where we assume \( E[\text{CCF}^{\text{ND}}_{t}] \) takes one of the three estimated values contingent on the expected real GDP growth rate at time \( t \). If time \( t \) is the default year, then the expected \( \text{EaD} \) is given by

\[
E[\text{EaD}_{t}] = E[U_{t-1}] + \text{CCF}^{\text{D}} \times (A_{t-1} - E[U_{t-1}])
= E[U_{t-1}] \times (1 - \text{CCF}^{\text{D}}) + A_{t-1} \times \text{CCF}^{\text{D}}, \tag{5.12}
\]

where we assume \( \text{CCF}^{\text{D}} \) is constant over time. In applying (5.11) and (5.12), we further assume \( A \) is deterministic over time. Suppose today is time \( t_0 \) and the current utilized and authorized dollar amounts of the facility are \( U_{t_0} \) and \( A_{t_0} \), respectively. Figure 3 illustrates the recursive calculations of the expected utilized amount and \( \text{EaD} \) term structure and the resulting EL over a three-year period.
Let us end our discussion with a simple numerical example. Suppose there is a line of credit for which we would like to calculate the EL over the next three years. The current utilized amount is $50,000. Suppose the authorized amount is fixed at $100,000 for the remaining life of the contract. The estimated CCF\textsuperscript{D} for the segment is 75%. There are three levels of CCF\textsuperscript{ND} estimated contingent on three different states of the economy: CCF\textsuperscript{ND} = 10% (high growth rate of real GDP), CCF\textsuperscript{ND} = 20% (moderate growth rate of real GDP) and CCF\textsuperscript{ND} = 40% (low growth rate of real GDP). Suppose that, consistent with CFCE, the economic outlook is such that the GDP growth rate is expected to be moderate in the upcoming year and then become slower in the subsequent years. Using (5.11) and (5.12), we calculate the expected utilized amounts and expected EaD for this line of credit for the next three years and report the results in Table 4.

Further, suppose the one-year conditional PD of the borrower is 5% for each of the next three years, and the expected LGD is always 50%. The three-year EL of this line of credit is therefore

\[
87,500 \times 50\% \times 5\% + 90,000 \times 50\% \times 5\% \times 95\% + 94,000 \times 50\% \times 5\% \times 95\% \times 95\% = 6446.
\]

### 5.3 Products with noncontractual maturities

Term to maturity is also an important variable in IFRS 9, considering that, in bucket 2, EL is estimated over the “life-time” of the loan. Our discussion in this paper is limited to loans with contractual maturities. There is also an extensive debate on how to deal with products with noncontractual maturities. An example would be credit cards. A-IRB calls for the use of a one-year term to maturity for retail products. This could be a result of banks’ argument that they maintain the option to cancel the credit card at any point in time if they are not satisfied with the creditworthiness of the cardholder. An “effective” term to maturity can be estimated from historical data for homogeneous segments of a given product. We can estimate the average card holding period, for example, for credit cardholders with a credit score over a certain threshold, which may be different (longer) than those with a lower credit score. An interesting

### TABLE 4  Expected utilized amounts and expected EaD over the next three years.

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E<a href="%25">CCF_{t}^{ND}</a>$</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>$E<a href="$">U_{t}</a>$</td>
<td>60,000</td>
<td>76,000</td>
</tr>
<tr>
<td>$E<a href="$">EaD_{t}</a>$</td>
<td>87,500</td>
<td>90,000</td>
</tr>
</tbody>
</table>
problem arises: when a bank estimates EL conditional on an expected macroeconomic outlook, not only PDs and LGDs (and EaD in certain cases), but also the effective term to maturity (thus the lifetime over which the EL will be calculated) will change with respect to the outlook.

Another product to consider is the revolvers, where there is an immediate maturity, upon arrival of which the loan can be extended (or renewed), provided that the conditions defined at origination are met. The question is whether the immediate maturity (the next renewal date) or the final maturity date (the maturity date if the loan is extended all the way) should be used as the effective maturity date. The implications are, of course, very significant, given that the next renewal date can be less than a year, whereas the final maturity date can be more than five or seven years. This is a topic currently being discussed by the BCBS. It is already accepted by the BCBS that if the loan is “unconditionally cancelable” by the bank at the next renewal date, the effective term to maturity is simply the next renewal date. The banks argue that this treatment be extended to loans that are “conditionally” cancelable by the bank. They argue that, should the borrower not meet the predefined performance targets (eg, meeting covenants, reaching a certain stage of development with the construction loans), the bank has the right to cancel the loan, and thus the next renewal date should be used as the effective maturity date rather than the final maturity date. The BCBS’s decision on the matter is expected over the next few months and the decision can be considered in the IFRS 9 debate as well. The banks have always had the option to estimate the effective maturity from historical data for homogeneous segments. This would likely produce an effective maturity date in between the next renewal date and the final maturity date.

The above argument would also hold for credit cards, as we expect that effective maturity will be conditional on the expected economic outlook for the revolvers as well. Certain behavioral characteristics can be observed from historical data for certain products and borrowers. For example, during the oil market downturn in Alberta (Canada), borrowers in the oil and gas servicing sector proactively sold their tangible assets and paid off their debt, and thus shortened their effective term to maturity. As a counterexample, obligors in other industries without such tangible assets can indeed try to extend the duration of the loan in order to survive. Some banks more readily exercise the conditional cancelations clause (and thus shorten the effective maturity) than others who find that the cancelation of the loan forces a default and actually increases the losses. This topic certainly warrants future study.

6 CONCLUSIONS

In this paper, we examine how we may utilize the A-IRB PD, LGD and EaD models for IFRS 9 and show how we can arrive at the expected one-year or lifetime credit
loss with the PD, LGD and EaD measures obtained from these models. We highlight the necessary model adaptations required to satisfy the new accounting standard of impairment measurement. By leveraging the A-IRB models, banks can lessen their modeling efforts in fulfilling IFRS 9 and capture the synergy among different modeling endeavors within the institutions. In introducing the proposed PD, LGD and EaD models, we provide detailed examples of how they may be implemented on secured lending.

To circumvent the need to assign probabilities to future scenarios in evaluating the expected PD in scenario analysis, we propose a convexity adjustment approach to deal with the nonlinear relation between conditional PD and the underlying macro-economic drivers. By doing so, we can enhance the objectivity and replicability of the resultant EL measure.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper. The opinions expressed in this paper are those of the authors and are not necessarily endorsed by the authors’ employers.

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