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LETTER FROM THE EDITOR-IN-CHIEF

Steve Satchell
Trinity College, University of Cambridge

This issue of *The Journal of Risk Model Validation* brings us four interesting papers, which I discuss below. I will resist the temptation to contextualize Brexit and risk model validation; much has been said already, much more will be said. Very loosely, two of our papers fall into the high-tech bucket while the other two fall into the practical guidance bucket (although one might be described as a hybrid); hopefully this leaves the journal offering something for everyone.

Our first paper, “Rating-transition-probability models and Comprehensive Capital Analysis and Review stress testing: methodologies and implementation” by Bill Hua-jian Yang and Zunwei Du, provides a new approach to the construction and validation of transition probability models. The paper explicitly demonstrates how Miu and Ozdemir’s original methodology on transition probability models can be structured and implemented with rating-specific asset correlation. A key innovation is to include a conditioning variable called the credit index. There are also applications based on a commercial portfolio illustrating estimation and scenario analyses.

The second paper, by Tae Yeon Kwon, is titled “A correlated structural credit risk model with random coefficients and its Bayesian estimation using stock and credit market information”. This paper illustrates a validation of a structural correlated default model applied to Black–Cox setups. While the dependence structure is modeled through the imposition of common factors on the asset process, instead of the assumption of homogeneity in the effects of common factors across the firms, a random coefficient representing the heterogeneity effect is considered. The approach taken is Bayesian; this finds few applications among practitioners, principally because of the difficulty of specifying priors that capture notions of skill or expertise in forecasting or risk management. Model parameters are estimated using not only equity prices but also credit default swap spreads. This led to improvements based on simulation relative to just using one of them. In order to demonstrate potential practical applications and an out-of-sample model validation check, the posterior distribution of CDX tranche prices is derived.

“Value-at-risk bounds for multivariate heavy tailed distribution: an application to the Glosten–Jagannathan–Runkle generalized autoregressive conditional heteroscedasticity model” by Imed Gammoudi, Mohamed El Ghourabi and Lotfi BelKacem is our third paper. It derives value-at-risk (VaR) bounds for the portfolios of possibly dependent financial assets for heavy tailed Glosten–Jagannathan–Runkle generalized autoregressive conditional heteroscedasticity processes using extreme value theory copulas. Based on earlier work by Mesfioui and Guess and by Gammoudi et al, the
authors show that these bounds have the interesting property of location invariance. As with all good theory papers, empirical studies for several market indexes are carried out to illustrate the authors’ approach.

The final paper in the issue, “Some options for evaluating significant deterioration under IFRS 9” by Gaurav Chawla, Lawrence R. Forest Jr. and Scott D. Aguais, addresses some issues to do with IFRS 9 (which, according to my rather limited understanding, is in the process of replacing IAS39). The point they focus on is, if I may quote them:

According to International Financial Reporting Standard 9 (IFRS 9), if the credit risk on an instrument has increased “significantly” since the instrument’s original recognition, and the resulting credit risk is more than “low credit risk”, then the institution would recognize a loss allowance on the instrument in the amount of lifetime expected credit losses (ECLs). Alternatively, at original recognition, and thereafter in the absence of significant deterioration in credit risk, the institution would recognize an allowance in the amount of twelve months of ECLs (or a lifetime, if the instrument matures in less than twelve months). In clarifying this aspect of IFRS 9, the International Accounting Standards Board (IASB) has specified that, in evaluating whether an instrument has suffered significant deterioration, an institution should consider only lifetime default risk, excluding consideration of possible changes in the exposure at default (EAD) and loss given default (LGD) components of ECLs. The authorities have also stated that the triggering of a lifetime allowance would reflect circumstances under which the spread inherent in contractual pricing no longer fully compensates for credit risk. Further, in its analysis of IFRS 9, the Bank for International Settlements has presented a view that any deterioration in credit risk should be considered significant.

Since many of the strictures in regulatory space are rather low on implementation detail, the authors go to some length to explain how to determine if an instrument has suffered serious deterioration in credit risk.

I will end on this note. In a recent editorial I mentioned reader feedback that had asked for guidance on responding to regulatory changes: it is a pleasure to be able to provide such guidance so soon.
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Research Paper

Rating-transition-probability models and Comprehensive Capital Analysis and Review stress testing: methodologies and implementation

Bill Huajian Yang and Zunwei Du

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ABSTRACT

Rating-transition-probability models, under the asymptotic single-risk-factor model framework, are widely used in the industry for stress testing and multi-period scenario loss projection. For a risk-rated portfolio, it is commonly believed that borrowers with higher risk ratings are more sensitive and vulnerable to adverse shocks. This means the asset correlation must be differentiated between ratings and fully reflected in all respects of model fitting. In this paper, we introduce a risk component, called the credit index, representing the systematic risk part of the portfolio by a list of macroeconomic variables. We show that the transition probability, conditional on this list of variables, can be formulated analytically by using the credit index and the rating-level sensitivity with respect to this credit index. Approaches are proposed for parameter estimation based on the maximum likelihood for observing historical rating transition frequency, in the presence of rating-level asset correlation. The proposed models and approaches are validated using a commercial portfolio, where the parameters for the conditional transition probability models are estimated, and the loss for baseline, adverse and...
severely adverse supervisory scenarios provided by the US Federal Reserve are projected for the period 2016 Q1–2018 Q1. This paper explicitly demonstrates how Miu and Ozdemir’s original methodology on transition probability models can be structured and implemented with rating-specific asset correlation. It extends our earlier work on this subject. We believe that the models and approaches proposed in this paper provide an effective tool to the practitioners for the use of transition probability models.

**Keywords:** CCAR stress testing; credit index; risk sensitivity; asset correlation; rating transition probability; maximum likelihood.

1 INTRODUCTION

The largest bank holding companies operating in the United States, with assets above US$10 billion, are subject to the Comprehensive Capital Analysis and Review (CCAR; Board of Governors of the Federal Reserve System 2016) annual exercise by the Federal Reserve to assess whether they have sufficient capital to continue operations throughout times of economic and financial stress, and whether they have robust, forward-looking capital-planning processes that account for their unique risks. The CCAR stress testing includes the assessment on a quarterly basis of loss on baseline, adverse and severely adverse scenarios, provided by the Federal Reserve and covering a period of nine quarters.

Under the advanced internal-ratings-based (AIRB) framework for a bank, a dynamic rating-transition-probability model provides a tool for multi-period scenario loss assessment: given the risk-rating distribution for a risk-rated portfolio at the beginning of a horizon, the risk-rating distribution at the end of the horizon can be derived by using the conditional transition probabilities given by a scenario. This calculation is reiterated over a period of time, and a loss projection for a multi-period scenario is thus obtained, given the exposure at default (EAD) and loss given default (LGD) components.

Let \( \{R_i \mid 1 \leq i \leq k\} \) denote a rating system with \( k \) ratings, where a smaller \( i \) indicates a lower default risk. Thus, \( R_1 \) is the highest quality rating and \( R_k \) is the worst rating, i.e., the default rating. It is assumed that, under the asymptotic single-risk-factor (ASRF) model framework, the risk for an entity with a non-default rating \( R_i \) is governed by a latent random variable \( z_i \), called the firm’s normalized asset value, which can be split into two parts, as follows (Basel Committee on Banking Supervision 2005; Gordy 2003; Merton 1974; Miu and Ozdemir 2009; Vasicek 2002; Yang and Du 2015):

\[
z_i = s \sqrt{\rho_i} + \varepsilon_i \sqrt{1-\rho_i}, \quad 0 < \rho_i < 1, \quad s \sim N(0, 1), \quad \varepsilon_i \sim N(0, 1), \tag{1.1}
\]
where \( s \) denotes the systematic risk (common to all non-default ratings) and \( \varepsilon_i \) is the idiosyncratic risk independent of \( s \). The quantity \( \rho_i \) is called the asset correlation for rating \( R_i \). It is assumed that there exist threshold values \( \{b_{ij}\} \) such that a firm’s rating migrates from \( R_i \) to \( R_j \) or worse (called a downgrade) within the horizon when \( z_i \) falls below the threshold value \( b_{i(k-j+1)} \).

The idiosyncratic risk can be factored into a transition probability model by the threshold values \( \{b_{ij}\} \). In contrast, modeling for systematic risk, in the presence of rating-level asset correlation, is challenging.

When the risk for a portfolio is believed to be homogeneous, uniform asset correlation can be assumed, and fitting for conditional transition probability models is relatively simple. However, for a risk-rated portfolio, it is commonly believed that borrowers with higher risk ratings are more sensitive and vulnerable to adverse shocks, and risk behaviors differ from rating to rating. This means the asset correlation needs to be differentiated between ratings and fully recognized in all respects of model fitting. This paper explicitly demonstrates how Miu and Ozdemir’s original methodology (Miu and Ozdemir 2009) on rating transition models can be structured and implemented with rating-specific asset correlation.

Rating-level asset correlation is recognized for transition probability models by Yang and Du (2015). However, the macroeconomic variable coefficients (ie, the coefficients in (a) below) are fitted via a regression (Yang and Du 2015, (3.1c)), without fully reflecting the rating-level asset correlation. We propose new approaches for parameter fitting, with rating-level asset correlation being fully recognized.

A credit index (CI), as introduced in the next section and in its simplest form, is a linear combination of a list of given macroeconomic variables, normalized to have zero mean and one standard deviation, under some appropriate assumption. As shown in Theorem 2.3, the conditional transition probabilities, given the list of macroeconomic variables, can be formulated analytically by the following three types of parameters:

(a) the coefficients of macroeconomic variables for the credit index;

(b) the risk sensitivity for a rating with respect to the credit index;

(c) the threshold values \( \{b_{ij}\} \).

Threshold values in (c) can be estimated separately (see Section 2.3). For the parameters in (a) and (b), we propose the estimation approaches by maximizing the log-likelihood of observing the historical rating transition frequency, with rating-level sensitivity fully incorporated.

The advantages of the proposed transition probability models and the parameter estimation approaches include the following.
(1) Rating-transition-probability models are structured by a credit index (representing the systematic risk part of the portfolio by a list of given macroeconomic variables) and the risk sensitivity with respect to this credit index for each rating.

(2) The proposed parameter estimation approaches are based on maximum likelihood of observing historical rating transition frequency. The rating-level asset correlation is fully recognized in all aspects of model fitting.

(3) Transition probability models fitted in this way are robust, not only at the portfolio level but also at the rating level.

The paper is organized as follows. In Section 2, we introduce the concept of the credit index, and show the analytical formulation of the conditional transition probability models, given a list of macroeconomic variables. In Section 3, we propose the parameter estimation approaches based on maximum likelihood of observing rating transition frequency. The proposed models and parameter estimation approaches are validated in Section 4, where we fit a transition probability model and project the loss for a commercial portfolio based on the supervisory scenarios provided by the Federal Reserve for the period 2016 Q1–2018 Q1.

2 RATING-TRANSITION PROBABILITY MODELS

In this section, we introduce the concept of credit index, a component representing the part of systematic risk explained by a given list of macroeconomic variables. We then show the analytical formulation of the condition rating transition probabilities, given a list of macroeconomic variables using this credit index.

2.1 Rating transition probabilities given the single latent systematic risk factor, s

Let $p_{ij}(s)$ denote the transition probability, given the single latent risk factor $s$, for a firm with a non-default risk rating $R_i$ at the beginning of a horizon and migrating to $R_j$ at the end of the horizon.

**Proposition 2.1** The following equations hold for the transition probability $p_{ij}(s)$:

$$p_{ij}(s) = \Phi\left[\frac{b_{i(k-j+1)} - s\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\right] - \Phi\left[\frac{b_{i(k-j)} - s\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\right]$$

$$= \Phi(\tilde{b}_{i(k-j+1)} - r_i s) - \Phi(\tilde{b}_{i(k-j)} - r_i s), \quad (2.1)$$
where

\[ r_i = \frac{\sqrt{p_i}}{\sqrt{1 - p_i}}, \quad (2.2) \]

\[ \tilde{b}_{ih} = \frac{b_{ih}}{\sqrt{1 - p_i}} = b_{ih}\sqrt{1 + r_i^2}. \quad (2.3) \]

**Proof** Expressions (2.1) and (2.2) follow from (1.1) and the definition of threshold values \( \{b_{ij}\} \). For (2.3), we have, by (2.2):

\[ \sqrt{1 + r_i^2} = \frac{1}{\sqrt{1 - p_i}} \quad \Rightarrow \quad \tilde{b}_{ih} = \frac{b_{ih}}{\sqrt{1 - p_i}} = b_{ih}\sqrt{1 + r_i^2}. \]

We call \( r_i \) the risk sensitivity for rating \( R_i \) with respect to the systematic risk factor \( s \).

By (2.1)–(2.3), the default probability \( p_{ik}(s) \) and downgrade probability \( p_{i(i+1)}(s) \) are given by

\[ p_{ik}(s) = \Phi(b_{i1}\sqrt{1 + r_i^2} - r_is), \quad p_{i(i+1)}(s) = \Phi(b_{i(k-i)}\sqrt{1 + r_i^2} - r_is). \]

Given a non-default rating \( R_i \), the risk sensitivity \( r_i \) can be estimated by maximizing the loglikelihood of observing the default or downgrade frequency, using, for example, SAS PROC NLMIXED (Yang and Du 2015).

### 2.2 The credit index and transition probabilities given macroeconomic variables

Given a list of macroeconomic variables \( x = (x_1, x_2, \ldots, x_m) \), let \( u_i \) be the mean value of \( x_i \). Consider a linear combination of the form

\[ c(x) = a_1x_1 + a_2x_2 + \cdots + a_mx_m. \]

Let \( \tilde{x}_i = (x_i - u_i) \). Normalize \( c(x) \) by setting

\[ \text{CI}(x) = \frac{c(x) - u}{v} = \frac{a_1\tilde{x}_1 + a_2\tilde{x}_2 + \cdots + a_m\tilde{x}_m}{v}, \]

where \( u \) and \( v \) respectively denote the mean and standard deviation of \( c(x) \). We assume that, in the presence of the given macroeconomic variables, the systematic risk factor \( s \) splits into two parts as follows:

\[ s = -\lambda \text{CI}(x) - e\sqrt{1 - \lambda^2} = -[\lambda(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + \cdots + \tilde{a}_m\tilde{x}_m) + \sigma e], \]

\[ e \sim N(0, 1), \quad 0 < \lambda < 1, \quad (2.4) \]
where
\[
\tilde{a}_i = \frac{a_i}{v}, \quad \sigma = \sqrt{1 - \lambda^2}.
\]

By (2.1)–(2.4), we have the following expressions for transition probability \(p_{ij}(s)\):

\[
p_{ij}(s) = \Phi[\tilde{b}_{i(k-j+1)} + r_i(\lambda C_1(x) + \sigma e)] \\
- \Phi[\tilde{b}_{i(k-j)} + r_i(\lambda C_1(x) + \sigma e)] \tag{2.5a}
\]

\[
= \Phi[\tilde{b}_{i(k-j+1)} + r_i(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + \cdots + \tilde{a}_m\tilde{x}_m) + r_i\sigma e] \\
- \Phi[\tilde{b}_{i(k-j)} + r_i(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + \cdots + \tilde{a}_m\tilde{x}_m) + r_i\sigma e]. \tag{2.5b}
\]

Let \(E_s[\Phi(a_0 + a_1s)]\) denote the expected value of \(\Phi(a_0 + a_1s)\) with respect to \(s\). In the subsequent discussions, we need the following lemma.

**Lemma 2.2** (Rosen and Saunders 2009)

\[
E_s[\Phi(a_0 + a_1s)] = \Phi \left( \frac{a_0}{\sqrt{1 + a_1^2}} \right),
\]

where \(s \sim N(0, 1)\).

Let \(p_{ij}(x) = E[p_{ij}(s) \mid x]\) be the expected value of transition probability \(p_{ij}(s)\) given macroeconomic variables \(x = (x_1, x_2, \ldots, x_m)\).

**Theorem 2.3** Assume that \(e\) in (2.4) is independent of \(x_1, x_2, \ldots, x_m\). Then the following equations hold for transition probability \(p_{ij}(x)\):

\[
p_{ij}(x) = \Phi(\tilde{b}_{i(k-j+1)} + \tilde{r}_i C_1(x)) - \Phi(\tilde{b}_{i(k-j)} + \tilde{r}_i C_1(x)) \tag{2.6a}
\]

\[
= \Phi(\tilde{b}_{i(k-j+1)} + \tilde{r}_i(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + \cdots + \tilde{a}_m\tilde{x}_m)) \\
- \Phi(\tilde{b}_{i(k-j)} + \tilde{r}_i(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + \cdots + \tilde{a}_m\tilde{x}_m)), \tag{2.6b}
\]

where

\[
\tilde{r}_i = \frac{r_i\lambda}{\sqrt{1 + r_i^2\sigma^2}} = \frac{r_i\lambda}{\sqrt{1 + r_i^2(1 - \lambda^2)}}, \tag{2.7}
\]

\[
\tilde{b}_{ih} = \frac{\tilde{b}_{ih}}{\sqrt{1 + r_i^2\sigma^2}} = b_{ih} \sqrt{1 + \tilde{r}_i^2}. \tag{2.8}
\]

**Proof** Expressions (2.6a) and (2.6b) follow from (2.5a) and (2.5b), respectively, by Lemma 2.2. For (2.7) and (2.8), we have, by Lemma 2.2 and (2.3),

\[
\tilde{b}_{ih} = \frac{b_{ih} \sqrt{1 + r_i^2}}{\sqrt{1 + r_i^2 (1 - \lambda^2)}}.
\]
By definition of \( \tilde{r}_i \), we have

\[
\tilde{r}_i = \frac{r_i \lambda}{\sqrt{1 + r_i^2 \sigma^2}} = \frac{r_i \lambda}{\sqrt{1 + r_i^2 (1 - \lambda^2)}} \quad \implies \quad \sqrt{1 + \tilde{r}_i^2} = \frac{\sqrt{1 + r_i^2}}{\sqrt{1 + r_i^2 (1 - \lambda^2)}}
\]

\[
\implies \hat{b}_{ih} = b_{ih} \sqrt{1 + \tilde{r}_i^2}.
\]

The linear combination \((\tilde{a}_1 \tilde{x}_1 + \tilde{a}_2 \tilde{x}_2 + \cdots + \tilde{a}_m \tilde{x}_m)\) in (2.6b) is constrained by

\[
v(\tilde{a}_1 \tilde{x}_1 + \tilde{a}_2 \tilde{x}_2 + \cdots + \tilde{a}_m \tilde{x}_m) = 1,
\]

where \(v(x)\) denotes the standard deviation of the random variable \(x\).

The default and downgrade probabilities have a simpler form and are respectively given by

\[
p_{i;k}(x) = \Phi\left[b_{i1} \sqrt{1 + \tilde{r}_i^2} + \tilde{r}_i (\tilde{a}_1 \tilde{x}_1 + \tilde{a}_2 \tilde{x}_2 + \cdots + \tilde{a}_m \tilde{x}_m)\right], \quad (2.10)
\]

\[
p_{i;(i+1)}(x) = \Phi\left[b_{i,k-i} \sqrt{1 + \tilde{r}_i^2} + \tilde{r}_i (\tilde{a}_1 \tilde{x}_1 + \tilde{a}_2 \tilde{x}_2 + \cdots + \tilde{a}_m \tilde{x}_m)\right]. \quad (2.11)
\]

Given the asset correlations \(\{\rho_i\}\) in (2.1) and thus \(\{r_i\}\), the risk sensitivity \(\tilde{r}_i\) with respect to \(CI(x)\) is driven by the parameter \(\lambda\) as in (2.7).

We define the credit index for a portfolio to be the \(CI(x)\), where the following conditions are satisfied.

(a) The residual \(e\) in (2.4) is independent of \(x_1, x_2, \ldots, x_m\).

(b) \(CI(x)\) is obtained from a normalization of a linear combination \(a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + \cdots + a_m \tilde{x}_m\) with which the model \(\{p_i(x)\}\) best predicts (through maximum likelihood as stated more specifically in Section 3.2) the default probability of the portfolio, where

\[
p_i(x) = \Phi[c_i + \tilde{r}_i (a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + \cdots + a_m \tilde{x}_m)]
\]

is a model predicting the default probability for the rating \(R_i\). The quantity \(\tilde{r}_i\) in (2.12) is driven by (2.7). No constraint is imposed for parameters \(\{c_i\}\) and \(\{a_1, a_2, \ldots, a_m\}\).

Condition (b) can be adapted to targeting the downgrade risk (rather than default risk) when the default rate for the portfolio is low.

Similarly to the quantities \(r_i\) and \(\rho_i\), defined under model (2.1) with respect to the single latent risk factor \(s\), we have the risk sensitivity \(\tilde{r}_i\) for rating \(R_i\) with respect to the credit index \(CI(x)\), and \(\tilde{\rho}_i\), which is defined as

\[
\tilde{\rho}_i = \rho_i \lambda^2.
\]
By using (2.1) and (2.4), we can think of $\tilde{\rho}_i$ as being the part of the asset correlation $\rho_i$ that is explained by the credit index CI(x).

**Proposition 2.4** For quantities $\tilde{r}_i$ and $\tilde{\rho}_i$, the following statements hold:

(a) similarly to expression (2.2), for $r_i$ and $\rho_i$,

$$\tilde{r}_i = \frac{\sqrt{\tilde{\rho}_i}}{\sqrt{1 - \tilde{\rho}_i}};$$

(b) $\tilde{r}_i < r_i$ and $\tilde{\rho}_i < \rho_i$.

**Proof** Statement (b) follows from $0 < \lambda < 1$ and $\sigma^2 = 1 - \lambda^2$. For (a), recall that

$$\tilde{r}_i = r_i \frac{\lambda}{\sqrt{1 + r_i^2 \sigma^2}} \quad \Rightarrow \quad 1 + \tilde{r}_i^2 = 1 + \frac{r_i^2 \lambda^2}{1 + r_i^2 \sigma^2} = \frac{1 + r_i^2}{1 + r_i^2 \sigma^2}$$

$$\quad \Rightarrow \quad \frac{\tilde{r}_i^2}{1 + \tilde{r}_i^2} = \frac{r_i^2 \lambda^2}{1 + r_i^2}.$$  \hspace{1cm} (2.13)

By (2.2), we have

$$r_i = \frac{\sqrt{\rho_i}}{\sqrt{1 - \rho_i}} \quad \Rightarrow \quad \rho_i = \frac{r_i^2}{1 + r_i^2}. \hspace{1cm} (2.14)$$

By (2.13) and (2.14), we have

$$\frac{\tilde{r}_i^2}{1 + \tilde{r}_i^2} = \rho_i \lambda^2 = \tilde{\rho}_i \quad \Rightarrow \quad \tilde{r}_i = \frac{\sqrt{\tilde{\rho}_i}}{\sqrt{1 - \tilde{\rho}_i}}.$$  \hspace{1cm} $\square$

Consequently, by (2.6a), for the determination of the transition probabilities \(\{p_{ij}(x)\}\), the following parameters (total \((k + 1)(k - 1) + m\)) are required:

(a) parameters $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$ for macroeconomic variables in credit index CI(x);

(b) rating-level risk sensitivities $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{k - 1}$;

(c) threshold values $\{b_{ij}\}$, $1 \leq i \leq k - 1$, $1 \leq j \leq k$.

**Remark 2.5** The threshold values $\{b_{ij}\}$ can be estimated separately, as shown in the next section. Therefore, the key to the transition probabilities $\{p_{ij}(x)\}$ is the determination of parameters $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$ and $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{k - 1}$.
2.3 Determination of the threshold values \( \{b_{ij}\} \)

Let \( p_{ij} = E[p_{ij}(s)] \) be the through-the-cycle (TTC) transition probability. By Lemma 2.2 and (2.1)–(2.3), we have

\[
p_{ij} = E\left[ \Phi(b_{i(k-j+1)} \sqrt{1 + r_i^2} - r_i s) \right] - E\left[ \Phi(b_{i(k-j)} \sqrt{1 + r_i^2} - r_i s) \right]
\]

\[
= \Phi(b_{i(k-j+1)}) - \Phi(b_{i(k-j)}).
\]

This means \( \{b_{ij}\} \) can be found by using the TTC transition probabilities \( \{p_{ij}\} \).

We now describe how the TTC transition probabilities \( \{p_{ij}\} \) can be determined by using the maximum likelihood approach.

Given \( i \), the loglikelihood of observing the rating transition frequencies \( \{n_{ij}\} \) (TTC frequencies are used here) is (up to a constant given by an appropriate multinomial coefficient number)

\[
\text{LL} = n_{i1} \log p_{i1} + n_{i2} \log p_{i2} + \cdots + n_{ik} \log p_{ik}.
\] (2.15)

Using the relation \( p_{ik} = 1 - p_{i1} - p_{i2} - \cdots - p_{ik-1} \) and setting to zero the derivative of (2.15) with respect to \( p_{ij}, 1 \leq j < k \), we have

\[
\frac{n_{ij}}{p_{ij}} - \frac{n_{ik}}{p_{ik}} = 0 \quad \Rightarrow \quad \frac{n_{ij}}{p_{ij}} = \frac{n_{ik}}{p_{ik}}.
\]

Because this holds for each \( j \) (\( 1 \leq j < k \)) for fixed \( k \), the vector \( (p_{i1}, p_{i2}, \ldots, p_{ik}) \) is proportional to \( (n_{i1}, n_{i2}, \ldots, n_{ik}) \). Therefore, the maximum likelihood estimate for \( p_{ij} \) is given by

\[
p_{ij} = \frac{n_{ij}}{n_{i1} + n_{i2} + \cdots + n_{ik}} = \frac{n_{ij}}{n_i},
\] (2.16)

where

\[
n_i = n_{i1} + n_{i2} + \cdots + n_{ik}.
\]

In general, we expect that a rating is more likely to migrate to a closer non-default rating than to a faraway non-default rating at the end of the horizon; and a higher risk rating carries a higher default probability. Therefore, the monotonicity constraints below are usually imposed:

\[
p_{ii+1} \geq p_{ii+2} \geq \cdots \geq p_{ik-1},
\]

\[
p_{i1} \leq p_{i2} \leq \cdots \leq p_{ii-1},
\]

\[
p_{1k} \leq p_{2k} \leq \cdots \leq p_{k-1k}.
\]

3 PROPOSED PARAMETER ESTIMATION APPROACHES

In this section, we propose the parameter estimation approaches based on maximum likelihood of observing historical rating transition frequency.
3.1 Loglikelihood functions for observing rating transition frequency

Given a non-default rating $R_i$ at the beginning of a horizon, we consider the following three rating transition frequencies.

(a) $n_{ij}$: the frequency transitioning to $R_j$ at the end of the horizon.

(b) $d_i$: the default frequency at the end of the horizon.

(c) $dd_i$: the frequency downgraded to a worse or default rating at the end of the horizon.

With the multinomial probability distribution, we have the following corresponding loglikelihood functions corresponding to the above transition frequencies (a)–(c), for all ratings for a single horizon (up to a constant independent of the parameters in $f_{p_{ij}}(x)$):

\[
LL = \sum_{j=1}^{k} n_{1j} \log(p_{1j}(x)) + \sum_{j=1}^{k} n_{2j} \log(p_{2j}(x)) + \cdots + \sum_{j=1}^{k} n_{(k-1)j} \log(p_{(k-1)j}(x)),
\]  
(3.1a)

\[
LL = \sum_{i=1}^{k-1} [(n_i - d_i) \log(1 - p_{ik}(x)) + d_i \log(p_{ik}(x))],
\]  
(3.1b)

\[
LL = \sum_{i=1}^{k-1} [(n_i - dd_i) \log(1 - p_{i(i+1)}(x)) + dd_i \log(p_{i(i+1)}(x))],
\]  
(3.1c)

where $n_i = n_{i1} + n_{i2} + \cdots + n_{ik}$.

For parameter fitting for the credit index, we use only (3.1b) (or (3.1c) when the default rate for the portfolio is low), with $p_{ik}(x)$ given by $p_i(x)$ in (2.12). For risk sensitivity fitting of \{$\tilde{r}_i$\}, we use (3.1b) or (3.1c), with $p_{ik}(x)$ or $p_{ik}(x)$ given by (2.6a) as follows:

\[
p_{ik}(x) = \Phi \left[ b_{i1} \sqrt{1 + \tilde{r}_i^2} + \tilde{r}_i CI(x) \right],
\]  
(3.2)

\[
p_{i(i+1)}(x) = \Phi \left[ b_{i(k-i)} \sqrt{1 + \tilde{r}_i^2} + \tilde{r}_i CI(x) \right].
\]  
(3.3)

The total loglikelihood for a time series sample is the sum of all the horizon loglikelihoods for all horizons.

A function is log concave if its logarithm is concave. If a function is concave, a local maximum is actually a global maximum, and the function is unimodal. This property is important for our maximum likelihood search.
**Proposition 3.1** The loglikelihood functions (3.1b) and (3.1c) are concave as a function of $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m$, and (3.2) or (3.3) is log concave as a function of $\hat{r}_i$. This concavity holds when the standard normal cumulative distribution $\Phi$ is replaced by any cumulative probability distribution that is log concave (eg, the cumulative distribution for a logistic distribution).

**Proof** It is well known that the standard normal cumulative distribution is log concave, and that the sum of concave functions is again concave. It is also known that if $f(x)$ is log concave, then so is $f(Az + b)$, where $Az + b: \mathbb{R}^m \to \mathbb{R}^1$ is any affine transformation from the $m$-dimensional Euclidean space to the one-dimensional Euclidean space. This means both the cumulative distribution $\Phi(x)$ and $F(x) = \Phi(-x)$ are log concave, and (3.1b) or (3.1c) is concave as a function of $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m$.

For the concavity of (3.2) or (3.3) as a function of $\hat{r}_i$, it suffices to show that the second derivative of the function

$$L(r) = \log \left[ \Phi(b \sqrt{1 + r^2} + ra) \right]$$

is nonpositive for any constants $a$ and $b$. The second derivative $d^2[L(r)]/dr^2$ is given by

$$\left( \frac{br}{\sqrt{1 + r^2}} + a \right)^2 \left\{ -\frac{[\varphi(b \sqrt{1 + r^2} + ra)]^2}{[\Phi(b \sqrt{1 + r^2} + ra)]^2} + \frac{\varphi'(b \sqrt{1 + r^2} + ra)}{\Phi(b \sqrt{1 + r^2} + ra)} \right\}$$

$$+ \frac{\varphi(b \sqrt{1 + r^2} + ra)(b)(1 + r^2)^{-3/2}}{\Phi(b \sqrt{1 + r^2} + ra)} = I + II,$$

where $\varphi$ and $\varphi'$ denote the first and second derivatives of $\Phi$. Because the factor in term I of (3.5),

$$\left\{ -\frac{[\varphi(b \sqrt{1 + r^2} + ra)]^2}{[\Phi(b \sqrt{1 + r^2} + ra)]^2} + \frac{\varphi'(b \sqrt{1 + r^2} + ra)}{\Phi(b \sqrt{1 + r^2} + ra)} \right\},$$

corresponds to a second derivative of log $\Phi(x)$, it is nonpositive. Thus, the first term in (3.5) is nonpositive. The second term in (3.5) is nonpositive if $b \leq 0$. For the case $b > 0$, we can change $b$ back to the negative case using the function $F(x) = \Phi(-x)$ and repeat the same discussion to obtain the nonpositivity of the second derivative of (3.4). \qed

### 3.2 Parameter estimation by maximum likelihood approaches

In this section, we assume that the threshold values $\{b_{ij}\}$ are known, and so are $\{r_i\}$, where $r_i$ is the risk sensitivity given by (2.2) for a non-default rating $R_i$ with
respect to the latent systematic risk factor $s$. This is because both $\{b_{ij}\}$ and $\{r_i\}$ are defined before observing any macroeconomic condition $x = (x_1, x_2, \ldots, x_m)$ (see Section 2.1 for the estimation of $\{r_i\}$, and Section 2.3 for $\{b_{ij}\}$).

As noted in Remark 2.5, the key to the rating transition probabilities $\{p_{ij}(x)\}$ is the determination of the coefficients $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$ for the credit index and the rating-level risk sensitivities $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{k-1}$. Recall that the credit index enters the model via (2.4) and is defined by parameters $\lambda, \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$. By Theorem 2.3, the following relation is satisfied for $\tilde{r}_i$:

$$
\tilde{r}_i = \frac{r_i \lambda}{\sqrt{1 + r_i^2 (1 - \lambda^2)}}. \tag{3.6}
$$

Given $\{b_{ij}\}$ and $\{r_i\}$, recall that the coefficients $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$ for the credit index are derived from a normalization of a linear combination $a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + \cdots + a_m \tilde{x}_m$, with which the model $\{p_i(x)\}$ best predicts the default probability of the portfolio, where $p_i(x)$ is given by (2.12) as

$$
p_i(x) = \Phi[c_i + \tilde{r}_i (a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + \cdots + a_m \tilde{x}_m)]. \tag{3.7}
$$

This can be implemented by using the loglikelihood function (3.1b), with $p_{ik}(x)$ being replaced by $p_i(x)$ above. Maximize the corresponding total loglikelihood for parameters $\lambda, \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$.

When $\lambda, \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$ are known, $\{\tilde{r}_i\}$ can in theory be determined by (3.5). However, for a better, more robust model, we propose to perform additional recalibration at the rating level for each $\tilde{r}_i$ by maximum likelihood using the total loglikelihood via (3.2) or (3.3) across time for that rating. The final rating transition model is given by (2.6a).

We thus propose the following two-step approach.

**Step 1: estimate $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$ for the credit index**

Randomly generate a number $\lambda \in (0, 1]$ and calculate $\{\tilde{r}_i\}$ using (3.5). Find the maximum likelihood estimates for $a_1, a_2, \ldots, a_m$ using the total loglikelihood from (3.1b) and (3.7) for all time for all ratings. By the concavity of (3.1b) as a function of $a_1, a_2, \ldots, a_m$, any of these local maximum likelihood estimates $a_1, a_2, \ldots, a_m$ are the global maximum likelihood estimates for a given $\lambda$. Use this list of parameter values as the initial values to maximize the likelihood again for these $(m + 1)$ parameters. Repeat this process sufficiently many times to obtain the global maximum likelihood estimate for $\lambda, a_1, a_2, \ldots, a_m$. Normalize the linear combination $a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + \cdots + a_m \tilde{x}_m$ to obtain the estimate for $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$. 
**Step 2: estimate \( \hat{r}_i \) for each non-default rating \( R_i \) separately**

Calculate credit index \( CI(x) \) as

\[
CI(x) = \frac{\hat{a}_1 x_1 + \hat{a}_2 x_2 + \cdots + \hat{a}_m x_m}{v},
\]

where \( v \) is the standard deviation of \( \hat{a}_1 x_1 + \hat{a}_2 x_2 + \cdots + \hat{a}_m x_m \). We then recalibrate and estimate \( \hat{r}_i \) by maximizing the total loglikelihood by (3.1b) or (3.1c) across time for rating \( R_i \), with \( p_{ik}(x) \) and \( p_{i(i+1)}(x) \) given by (2.6a) as

\[
p_{ik}(x) = \Phi \left[ b_{i1} \sqrt{1 + \hat{r}_i^2} + \hat{r}_i CI(x) \right],
\]

\[
p_{i(i+1)}(x) = \Phi \left[ b_{ik-i} \sqrt{1 + \hat{r}_i^2} + \hat{r}_i CI(x) \right].
\]

We implemented the above two-step optimization process by using the SAS PROC NLMIXED procedure.

### 4 AN EMPIRICAL EXAMPLE: COMPREHENSIVE CAPITAL ANALYSIS AND REVIEW STRESS TESTING FOR A COMMERCIAL PORTFOLIO

In this section, we fit the rating transition model for a commercial portfolio, and assess the nine quarter losses for the portfolio for the supervisory scenarios provided by the Federal Reserve for the period 2016 Q1–2018 Q1.

The data is created synthetically from the historical quarterly rating transition frequency for a US commercial portfolio (the sample default rate does not reflect the original portfolio’s true default rate). There are seven ratings for the portfolio, with rating \( R_1 \) being the best quality rating and \( R_7 \) as the default rating.

We match the sample to the macroeconomic data (sourced from the Federal Reserve) by calendar quarter. We are focused on the following nine macroeconomic variables given in Table 1. The selection of variables is subject to a governance review process. Each variable should pass the unit root tests. Here, we consider four lag variables for each macroeconomic variable: lag 0 (current), lag 1 (lag one quarter), lag 2 (lag two quarters), lag 3 (lag three quarters). Each lag variable is named by prefixing the original name with a label “L” together with its lag number.

In the remainder of this section, we focus on model fitting and scenario loss projection.

#### 4.1 Variable selection

Let \( m \) denote the number of variables in a model. Due to the limited number of data points in the time series sample, we consider only models with \( m \leq 4 \). A preliminary
TABLE 1 Macroeconomic variables.

<table>
<thead>
<tr>
<th>#</th>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>GDP_GQOQ_COM</td>
<td>Growth rate of US gross domestic product (quarter over quarter, annualized by compounding)</td>
</tr>
<tr>
<td>2</td>
<td>LURC_DQOQ</td>
<td>Increase in US civilian unemployment rate (quarter over quarter, annualized)</td>
</tr>
<tr>
<td>3</td>
<td>PCREPI_GQOQ_COM</td>
<td>Growth rate of US commercial real estate price (quarter over quarter, annualized by compounding)</td>
</tr>
<tr>
<td>4</td>
<td>PPSDJT_GQOQ_COM</td>
<td>Growth rate of Dow Jones Total Stock Market Index (quarter over quarter, annualized by compounding)</td>
</tr>
<tr>
<td>5</td>
<td>RCBBB_DQOQ</td>
<td>Increase in US BBB ten-year corporate yield (quarter over quarter annualized)</td>
</tr>
<tr>
<td>6</td>
<td>RCBBB_RT10Y</td>
<td>US ten-year BBB corporate credit spread</td>
</tr>
<tr>
<td>7</td>
<td>RTB_DQOQ</td>
<td>Increase in US three-month treasury bill: secondary market rate (quarter over quarter, annualized)</td>
</tr>
<tr>
<td>8</td>
<td>RT10Y_DQOQ</td>
<td>Increase in US Constant Maturity Treasury Yield, ten-year (quarter over quarter, annualized)</td>
</tr>
<tr>
<td>9</td>
<td>VIX_FED</td>
<td>US implied volatility (maximum of daily values per quarter)</td>
</tr>
</tbody>
</table>

The model-selection process is performed via SAS logistic regression with the model selection option being set to “Score”, targeting portfolio default frequency over the sample. The best 1000 models (in the form of a variable combination, no coefficients are provided by SAS with this model selection option) for each value of $m$ are selected for subsequent evaluations.

4.2 Transition probability model fitting

For each list of macroeconomic variables $x_1, x_2, \ldots, x_m$ from Section 4.1, follow the steps proposed in Section 3.2 to fit for coefficients $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$ and sensitivities $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{k-1}$.

Table 2 shows the top ten transition probability models based on their accuracy in predicting the portfolio default rate in the downturn period 2008 Q1–2009 Q4, where the bottom part of the table gives the risk sensitivity for each non-default rating with respect to the corresponding credit index. The column “MAD” denotes the average deviation (average of absolute values for the prediction error) between the realized and predicted portfolio default rates over the period 2008 Q1–2009 Q4.
### TABLE 2  Top ten models.

<table>
<thead>
<tr>
<th>Model #</th>
<th>CI model variable 1</th>
<th>CI model variable 2</th>
<th>CI model variable 3</th>
</tr>
</thead>
<tbody>
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<td>L1_RT10Y_DQOQ</td>
<td></td>
</tr>
<tr>
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<tr>
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<tr>
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<tr>
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</tr>
<tr>
<td>8</td>
<td>L3_PCREPI_GQOQ_COM</td>
<td>L0_LURC_DQOQ</td>
<td>L2_LURC_DQOQ</td>
</tr>
<tr>
<td>9</td>
<td>L1_RCBBB_RT10Y</td>
<td>L0_PPSDJT_GQOQ_COM</td>
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<tr>
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<table>
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<tr>
<th>Model #</th>
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<th>Parm2</th>
<th>Parm3</th>
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<table>
<thead>
<tr>
<th>Model #</th>
<th>Rating level risk sensitivity to CI 1</th>
<th>Rating level risk sensitivity to CI 2</th>
<th>Rating level risk sensitivity to CI 3</th>
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<th>Rating level risk sensitivity to CI 5</th>
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4.3 Transition probability model performance

Figure 1 shows the backtest results for the top transition model (#1 in Table 2) between 2006 Q1 and 2015 Q1. In general, we expect the predicted default rate to be higher in the downturn period 2008 Q1–2010 Q1.

Parts (a) and (b) show the predicted and realized default rates at the portfolio level and for rating $R_6$, respectively. It turns out that this model is able to pick up the default rate at the portfolio level and also for ratings $R_3$, $R_4$, $R_5$ and $R_6$. For rating $R_1$, the best quality rating, the predicted default rate is flat, as expected, due to its low realized default rate (close to zero except for one quarter) for this rating.

4.4 Scenario loss projection

Let $s_1(t_0), s_2(t_0), \ldots, s_7(t_0)$ denote the percentage distribution for ratings $R_1, R_2, \ldots, R_7$ for the portfolio at $t_0$, the beginning of a horizon. Let $\{p_{ij}(x)\}$ be transition probabilities for this horizon given the list of macroeconomic variables $x = (x_1, x_2, \ldots, x_m)$. To facilitate subsequent calculations, we add another row, $\{p_{7j}\}$, to the transition matrix: $p_{7j}(x) = 0$ for $1 \leq j \leq 6$, and $p_{77}(x) = 1$. Then the rating distribution for the portfolio at $t_1$, the end of the horizon, is given by

$$s_i(t_1) = s_1(t_0) p_{1i}(x) + s_2(t_0) p_{2i}(x) + \cdots + s_7(t_0) p_{7i}(x).$$

Let $l_i(t_1)$ and $e_i(t_1)$ respectively denote the LGD and EAD factors for a default facility at $t_1$ for risk rating $R_i$. Then the marginal portfolio default rate and marginal loss over the period $(t_0, t_1]$ are respectively given by

$$p_{t_1} = s_1(t_0) p_{17}(x) + s_2(t_0) p_{27}(x) + \cdots + s_6(t_0) p_{67}(x),$$

$$L(t_1) = s_1(t_0) p_{17}(x) f_1(t_1) + s_2(t_0) p_{27}(x) f_2(t_1) + \cdots + s_6(t_0) p_{67}(x) f_6(t_1),$$

where $f_i(t_1)$ is the sum of products $[l_i(t_1)][e_i(t_1)]$ over all facilities for the borrower.

Using the top transition model selected from Table 2, in Table 3 we calculate the portfolio default rate and loss for each quarter for baseline, adverse and severely adverse scenarios, provided by the Fed for a period of nine quarters from 2016 Q1 to 2018 Q1. Here, the cumulative portfolio default rate $c_t$ at time $t$ is calculated from the marginal default rate $p_t$ by using the formula

$$c_t = c_{t-1} + (1 - c_{t-1}) p_t.$$

The loss is presented as a percentage of the portfolio total exposure at the beginning of the period. The results show that the model projects a loss of 3.42% for the baseline scenario, 4.24% for the adverse scenario and 6.11% for the severely adverse scenario.
FIGURE 1  Predicted versus realized portfolio default rate for the top model.

(a) Portfolio. (b) $R_6$. (c) $R_5$. (d) $R_4$. (e) $R_3$. (f) $R_2$. (g) $R_1$. Dashed line shows predicted rate. Solid line shows realized rate.

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5 CONCLUSIONS

Rating-transition-probability models are widely used in the industry for multi-period scenario loss projection. We have demonstrated explicitly how Miu and Ozdemir’s original methodology (Miu and Ozdemir 2009) can be structured and implemented with rating-specific asset correlation. The models we proposed are structured by using a risk component, called the credit index, representing the part of systematic risk for the portfolio explained by a list of macroeconomic variables. Rating transition probabilities were formulated analytically by using the rating-level sensitivity with respect to this credit index. The proposed parameter estimation approaches are based on maximum likelihood for observing the historical rating transition frequency. Rating-level asset correlation is fully recognized in all respects of model fitting. The resulting
models obtained by using these approaches are in general robust not just at the portfolio level but also at the rating level. These approaches can be implemented easily by modelers, using, for example, SAS PROC NLMIXED (Yang and Du 2015). We believe that the models and approaches proposed in this paper provide an effective tool for practitioners when using migration matrix methodology for CCAR stress testing and loss projection.

DECLARATION OF INTEREST

The views expressed in this paper are not necessarily those of the Royal Bank of Canada or any of its affiliates. The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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A correlated structural credit risk model with random coefficients and its Bayesian estimation using stock and credit market information

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ABSTRACT

Using historical equity and credit market data, we illustrate the validation of a structural correlated default model applied to Black–Cox setups. We model the dependence structure through the imposition of common factors on the asset process. Instead of assuming homogeneity in the effects of the common factor across the firms, we consider a random coefficient representing the heterogeneity effect. Based on the Bayesian method, we estimate model parameters using not only equity prices but also credit default swap (CDS) spreads. Through our simulation studies, we found that the estimation performance improved when both stock prices and CDS spreads were used compared with the use of stock prices alone. Our empirical analysis is based on daily data for the 125 issuers comprising the CDS.NA.IG13 in 2009. In order to demonstrate potential practical applications and check the out-of-sample model validation, we derive the posterior distribution of CDX tranche prices.

Keywords: default correlation; random coefficient model; credit default swap (CDS); Black–Cox structural model; generalized Gibbs sampling.
1 INTRODUCTION

Since Black and Scholes (1973) and Merton (1973) first considered equity as a call option on the firm’s asset value, Merton’s structural model has remained the basic reference for pricing defaultable single bonds. In terms of the multi-firm structural model, Zhou (2001) examined the structural model, which is extended to two firms whose assets are correlated, and found a closed-form formula for the joint default probability of two issuers. However, extending Zhou’s results to more than two issuers is difficult. A method to handle the correlation between a large number of firms by imposing a factor structure on default intensities has been studied in Duffie and Garleanu (2001), Duffie et al (2007) and Duffie et al (2009).

In a manner similar to that of the common factor method in the typical intensity models, Hull et al (2010) provided a way to model correlation under the structural model. They divided the Wiener process governing asset $W_{it}$ into a common component $X_t$ and an idiosyncratic component $W_{\epsilon_{it}}$ and then measured the degree of sensitivity to the common factor using $a_i$, as follows:

$$dW_{\epsilon_{it}} = a_i dX_t + \sqrt{1 - a_i^2} dW_{\epsilon_{it}}. \quad (1.1)$$

With these $a_i$, they measured asset correlation. They tested how well the model fitted the real market using the CDX North American Investment Grade index (CDX.NA.IG) and iTraxx Europe data.

In this paper, we apply the structural correlated default model of Hull et al (2010) with several extensions to the Black and Cox (1976) structural model instead of the Merton model. Even though its pricing function is more complicated than that of the Merton model, the Black–Cox model has a default assumption that is more relaxed and closer to the real world, in that default can occur at any time prior to the maturity of the bond.

We model the correlation of defaults using correlated asset processes sharing a common factor process $X_t$, as in (1.1). However, we allow the degree of dependence on common factors, which is represented by $a_i$ in (1.1), to differ and be random. By assuming the coefficients $a_i$ are random, two extreme assumptions about $a_i$ can be compromised: strictly different common-factor effects (ie, $a_i \neq a_j$ for all $i, j$) and equal common factor effects across firms (ie, $a_i = a_j$ for all $i, j$). A probability distribution is assigned to the coefficients and we estimate not only all $a_i$s, but also the hyperparameters governing the $a_i$s’ probability distribution. Hyperparameters of random coefficients make it possible for us to measure the degree of homogeneity in the firm-specific common-factor effect and the overall magnitude of the corresponding common-factor effect.
Hull et al (2010) used fixed $a_i$ values and tested their adequacy. We estimate the model parameters and asset values using a Bayesian method; Gibbs sampling and the Metropolis–Hastings (MH) algorithm are the basic methods used. However, these methods are used with the simultaneous jump technique to improve chain convergence (Kou et al 2005; Liu 1996).

Rather than simply adopting Bayesian schemes for parameter and asset value estimation, we approach estimation problems in terms of cross-asset-class research. Credit default swap (CDS) information is additionally used in order to improve the estimation performance, while standard practice suggests estimating asset values from stock prices only. Such cross-asset-class research, viewing the firm’s capital structure through multiple classes of securities, has already gained attention in other fields of financial research in recent years. Ni et al (2008) and Johannes et al (2009) illustrated that using option prices gives more efficient and accurate results in stock return volatility estimation than using equity prices alone. However, Kwon and Lee (2016) showed that option prices only marginally improve estimation efficiency in the credit risk model. Hence, in this paper we add CDS information to further improve such estimation efficiency.

In order to validate the model estimation process, we conduct a simulation study. The method of parameter estimation and the incorporated market data are evaluated based on hypothetical data. Based on the simulation results, we can confirm that the Bayesian method adopted in this paper estimates all model parameters accurately. The simulation results also show that CDS prices enhance the precision of the estimation to a greater extent than using stock market information alone.

The empirical results are derived from the daily equity and CDS prices of the 125 companies constituting the CDX.NA.IG13 in 2009. Results showing that common-factor coefficients vary widely between different firms validate the need to relax the assumption of equal factor coefficients. Based on the estimated parameters and asset values, we can predict the prices of CDX tranche. There is a rich literature on CDX tranche pricing. Longstaff and Rajan (2008) and Jobst et al (2015), in particular, also adopt random-factor approaches. However, the prediction of CDX tranche prices is also meaningful as a model validation check using out-of-sample data, because the CDX tranche price data is not incorporated within the estimation procedure in this paper.

The paper is constructed as follows. Section 2 introduces our default correlation model. Section 3 provides details of the Bayesian estimation procedure, adopting the selective simultaneous jump scheme in the Gibbs sampling. Based on simulated data, Section 4 investigates the performance of the proposed Bayesian estimation method and the additional information CDS can provide. Section 5 presents the empirical results. In Section 6, the posterior distribution of CDX tranche prices is derived. The conclusions are presented in Section 7.
2 MODEL

Because the asset process is not observable, it can be inferred through observed financial market prices such as stock and CDS prices. Whereas standard methods directly calculate asset values using the inverse function of the stock pricing formula, this paper adopts the state-space model in order to combine different pieces of market information. The unobserved asset is set to be a state variable and the CDS and stock prices are set as observation equations.

State equation:

\[ \frac{dV_{it}}{dt} = \mu_i V_{it} dt + \sigma_i V_{it} dW^V_{it}, \quad (2.1) \]

\[ dW^V_{it} = a_{1i} dX_{1t} + a_{2i} dX_{2t} + a_{3i} dX_{3t} + \sqrt{1 - a_{1i}^2 - a_{2i}^2 - a_{3i}^2} dW^e_{it}. \quad (2.2) \]

Observation equation:

\[ \ln S_{it} = g_S(V_{it}, \sigma_i, T_{Di}, D_i) + Z^S_{it}, \quad Z^S_{it} \sim N(0, \sigma_S), \quad (2.3) \]

\[ \ln \text{CDS}_{it} = g_{\text{CDS}}(V_{it}, \sigma_i, T_{\text{CDS}i}) + Z^\text{CDS}_{it}, \quad Z^\text{CDS}_{it} \sim N(0, \sigma_{\text{CDS}}). \quad (2.4) \]

\( N \) different companies are considered. Index \( i = 1, \ldots, N \) is used for the companies, and index \( t = 1, \ldots, T \) for the time index. \( V_{it} \) refers to the \( i \)th company’s asset value at time \( t \). \( dX_{1t}, dX_{2t} \) and \( dX_{3t} \) are observed common-factor processes, which are assumed to follow a Brownian motion. \( S_{it} \) is the observed equity price of the \( i \)th firm at time \( t \), and \( \text{CDS}_{it} \) is the observed CDS spread of the \( i \)th firm at time \( t \). \( Z^S_{it} \) and \( Z^{\text{CDS}}_{it} \) are stock and CDS market noise, respectively. \( T_{Di} \) and \( T_{\text{CDS}i} \) are the maturity of the bond and CDS, respectively. \( D_i \) is the face value of the bond. \( g_S(\cdot) \) and \( g_{\text{CDS}}(\cdot) \) represent model-derived equity and CDS values. They are both known functions referred to in Lando (2004), Zhou (2001) and Black and Cox (1976) and are given in online Appendixes A and B, respectively.

All coefficients \( a_{1i}, a_{2i} \) and \( a_{3i} \) for \( i = 1, \ldots, N \) are assumed to be random, and their distribution function \( P(a_{1i}, a_{2i}, a_{3i}) \) is given by

\[ P(a_{1i}, a_{2i}, a_{3i}) \propto \exp \left\{ -\frac{(a_{1i} - \mu_{a1})^2}{2\sigma_{a1}^2} \right\} I_{-1 \leq a_{1i} \leq 1} \times \exp \left\{ -\frac{(a_{2i} - \mu_{a2})^2}{2\sigma_{a2}^2} \right\} I_{-1 \leq a_{2i} \leq 1} \times \exp \left\{ -\frac{(a_{3i} - \mu_{a3})^2}{2\sigma_{a3}^2} \right\} I_{-1 \leq a_{3i} \leq 1} \times I_{a_{1i}^2 + a_{2i}^2 + a_{3i}^2 \leq 1}. \quad (2.5) \]

Because \( a_{1i}, a_{2i} \) and \( a_{3i} \) are all bounded and jointly constrained due to the square-root term in (2.2), truncated normal distributions with a joint constraint for \( a_{1i}, a_{2i} \)
and \( a_{3i} \) form their joint probability distribution. \( \mu_{a_1}, \sigma_{a_1}, \mu_{a_2}, \sigma_{a_2}, \mu_{a_3} \) and \( \sigma_{a_3} \) are also all estimated as hyperparameters in the hierarchical model. For all \( \mu_i, \sigma_i \) and the hyperparameters, we assign noninformative priors.

### 2.1 Model details

In this section, we provide further details about the common factors, random coefficients and cross-asset-class approach taken in our correlated structural model.

#### 2.1.1 Common factors

Asset value \( V_{it} \) is assumed to follow geometric Brownian motion under objective measure \( P \) as in (2.1). The dependence structure between firms is modeled by the correlated standard Brownian motion \( W^Y_{it} \) as in (2.2). Through the shared use of \( dX_{1t}, dX_{2t}, \) and \( dX_{3t} \), which are Brownian motions underlying the common factors, the dependence of the asset value processes of different firms is modeled.\(^1\) Using correlated standard Brownian motions in a structural model in this way is suggested in Hull et al (2010).

In order to choose \( X_1, X_2 \) and \( X_3 \), we refer to studies on stock return predictability. The various estimation methods and the predictive power of financial variables are covered in Campbell and Yogo (2006) and Fama and French (1996a,b). Even though there are differences in the names and forms of the variables, the market return, value premium and size premium are the most commonly used factors.

The following three variables are considered in this paper. As in previous studies, any interaction effects between variables are not included.

1. **Market return (SP500):** log return of S&P 500 \((\ln(S&P\ 500_t/S&P\ 500_{t-1}))\).
2. **Value premium (HML):** the difference between the return on a portfolio of high-book-to-market stocks and the return on a portfolio of low-book-to-market stocks. This factor is applied after log-transformation as \( \ln(1 + \text{HML}) \).
3. **Size premium (SMB):** the difference between the return on a portfolio of small stocks and the return on a portfolio of large stocks. This factor is applied after log-transformation as \( \ln(1 + \text{SMB}) \).

All three of these variables are easily observable in the financial market. The market return represents the general economic condition, while the HML is known as a proxy for relative distress. The SMB represents the small stocks, which are not captured by the average market return.

In applying the observed common factor, we use Brownian motions underlying the evolution of the corresponding observed common factors. The volatilities of \( X_{1t}, \)

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\(^1\) Instead of using return values, we transform all observed common factors into price values.
$X_{2t}$ and $X_{3t}$ are all standardized because they can be measured with each coefficient term $a_{1i}$, $a_{2i}$ and $a_{3i}$, respectively, in (2.2). All common factors are assumed to be independent.

### 2.1.2 Random coefficients

By adopting a random coefficient model, we allow a degree of dependence in the common factors, represented by $a_{1i}$, $a_{2i}$ and $a_{3i}$ in (2.2), which can differ by firm. The random coefficient idea makes a compromise between the following two extreme assumptions:

1. general economic conditions affect all firms equally (ie, $a_i = a_j$ for all $i, j$),

and

2. general economic conditions affect all firms differently (ie, $a_i \neq a_j$ for all $i, j$).

By adopting random coefficients and assigning a probability distribution to them, the results of individual firms can be partially pooled (Gelman and Hill 2007).

The hyperparameters determining the distribution of $\mu_{a_1}, \sigma_{a_1}, \mu_{a_2}, \sigma_{a_2}, \mu_{a_3}$ and $\sigma_{a_3}$ in (2.5) are also estimated from the data using the Bayesian method. The priors for the hyperparameters (the hyperpriors) are assumed to be flat and noninformative.

The choice of normal prior for $a_{1i}, a_{2i}$ and $a_{3i}$ as in (2.5) is made because of the advantages it provides for both interpretation and statistical estimation. In terms of interpretation, Koop et al (2008) and Gelman and Hill (2007) pointed out that the normal prior provides a soft homogeneity and soft constraint to parameters. They also address how the degree of constraint or homogeneity can be adjusted using the variances of the normal priors, $\sigma_{a_1}^2$, $\sigma_{a_2}^2$ and $\sigma_{a_3}^2$. In the limit $\sigma_{a_i}^2 \to \infty$, the soft constraints do nothing, and there is no pooling or complete heterogeneity; as $\sigma_{a_i}^2 \to 0$, they pull the estimates all the way to zero, yielding complete pooling or complete homogeneity estimates. In our hierarchical model, these variances of the priors are also estimated based on data, and we assume noninformative priors for the hyperparameters. Therefore, we see no reason to accept estimates that arbitrarily set this parameter to one of these two extremes ($0$ or $\infty$).

Moreover, in terms of statistical estimation, a normal prior simplifies the form of the posterior distribution because the likelihoods of the data, $\ln S$ and $\ln$ CDS, are also both normal. Hence, the normal distribution is a favorable choice and is also commonly used for priors. McNeil and Wendin (2007) and Koop et al (2008) adopt normal priors for their Bayesian multilevel regression analysis with financial data.
2.1.3 Cross-asset-classes approach: equity and CDS

Another distinctive approach in this paper is that both equity and CDS prices are used together in the observation equation, as in (2.3) and (2.4). For the error between the model and market values, $Z_{Si}$ and $Z_{CDSi}$, a normal distribution for both stock and CDS is assumed, as in Duan and Fulop (2009) and Pan and Singleton (2008). $\sigma_S$ and $\sigma_{CDS}$ are assumed to be known.

The standard deviation of stock market noise $\sigma_S$ is set to be 0.01 as in Kwon and Lee (2016). This is a generous assumption in that, according to Duan and Fulop (2009), 95% of the differences between the market and theoretical equity values for Dow Jones 30 companies are in the range ±1.46%.

A larger variance is set for the CDS market than for the stock market, due to the difference in market liquidity. In order to compare market liquidity, bid–ask spreads are usually used because it can be assumed that the correlation coefficient between the bid–ask spread and market noise is positive (Aït-Sahalia and Jialin 2009). During 2009, the average CDS bid–ask spread of the 125 companies constituting the CDX.NA.IG13 was 10.4 basis points. The average stock-price bid–ask spread of the same 125 companies was 0.87. As Pan and Singleton (2008) assumed a different standard deviation of CDS market noise, setting it as a value proportional to their market bid–ask spreads, we set $\sigma_{CDS}$ to be 0.1, which is ten times the value of $\sigma_S$.

Even though additional assumptions about the noise terms are necessary, including market noise terms $Z_{Si}$ and $Z_{CDSi}$ in a model has two implications. First, market noise makes it possible for us to combine the information from several different markets in estimating the asset-values. Second, market noise captures the model misspecification error or short-term discrepancies in the supply and demand within financial markets. More discussion on market noise and stock market model misspecification errors is provided in Kwon and Lee (2016).

2.2 Properties of the model

In summary, the correlated structural model in this paper is the following state space model.

**State equation:**

$$d \ln V_{it} = \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) dt + \sigma_i \left(a_{1i} dX_{1t} + a_{2i} dX_{2t} + a_{3i} dY_t \right) + \sigma_i \sqrt{1 - a_{1i}^2 - a_{2i}^2 - a_{3i}^2} dW_{it}^e.$$

**Observation equation:**

$$\ln S_{it} = g_S(V_{it}, \sigma_i, T_D, D) + Z_{Si}, \quad (2.6)$$
$$\ln CDS_{it} = g_{CDS}(V_{it}, \sigma_i, T_{CDS}) + Z_{CDSi}. \quad (2.7)$$
The state equation is a hierarchical linear random coefficient model. At the observation level, equity and CDS prices are all incorporated under the Black–Cox model, by allowing model misspecification error.

For the feasibility of analysis, the model is analyzed in discretized time, with \( \Delta t = 1/252 \). In a discrete-time framework, the conditional distributions \( P_{V_t} (\ln V_{it} |\ln V_{it-1}, X_{1:t}, \Theta_V) \) for all \( i \) and \( t \) are assumed to be independent and identically distributed normal distributions, with mean \( \mu_{V_{it},V_{it-1}} \) and standard deviation \( \sigma_{V_{it},V_{it-1}} \) as follows:

\[
\begin{align*}
\mu_{V_{it},V_{it-1}} &= \ln V_{it-1} + (\mu_i - \frac{1}{2} \sigma_i^2) \Delta t \\
&\quad + \sigma_i (a_{1i}(\Delta X_{1t}) + a_{2i}(\Delta X_{2t}) + a_{3i}(\Delta X_{3t})), \quad (2.8) \\
\sigma_{V_{it},V_{it-1}} &= \sigma_i \sqrt{1 - a_{1i}^2 - a_{2i}^2 - a_{3i}^2} \Delta t, \quad (2.9)
\end{align*}
\]

where the parameter vector \( \Theta_V \) includes all random coefficients \( (\mu_1, \ldots, \mu_N, \sigma_1, \ldots, \sigma_N, a_{11}, \ldots, a_{1N}, a_{21}, \ldots, a_{2N}, a_{31}, \ldots, a_{3N}) \) and their hyperparameters \( (\mu_{a1}, \sigma_{a1}, \mu_{a2}, \sigma_{a2}, \mu_{a3}, \sigma_{a3}) \).

One of the advantages of this model is that asset process volatility can be decomposed into common and idiosyncratic parts.

- **Common part:** \( \sigma_i \sqrt{a_{1i}^2 + a_{2i}^2 + a_{3i}^2} \).
- **Idiosyncratic part:** \( \sigma_i \sqrt{1 - a_{1i}^2 - a_{2i}^2 - a_{3i}^2} \).

Furthermore, through this model, the correlation between the \( i \)th and \( j \)th firms can be measured using the coefficients \( a_1, a_2 \) and \( a_3 \) as follows:

\[
\text{Corr}(d \ln V_{it}, d \ln V_{jt} | a_1, a_2, b, \sigma_i, \sigma_j) = a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j}. \quad (2.10)
\]

### 3 ESTIMATION

The basic statistical tools used for parameter estimation are Gibbs sampling and the MH algorithm. The generalized Gibbs sampler introduced in Liu and Sabatti (2000) and Kou et al (2005) is adopted in this paper, with a modification for improved chain convergence.

The parameter vector to be estimated is

\[
\Theta_V = (\mu_1, \ldots, \mu_N, \sigma_1, \ldots, \sigma_N, a_{11}, \ldots, a_{1N}, a_{21}, \ldots, a_{2N}, a_{31}, \ldots, a_{3N}, \\
\mu_{a1}, \sigma_{a1}, \mu_{a2}, \sigma_{a2}, \mu_{a3}, \sigma_{a3})
\]
and \( \ln V = \ln V_{1:1:T} \) are also unknown. Given that all \( \ln V_{i0} \) are known, the conditional distribution of unknown values \( P(\ln V, Y, \Theta_Y | S, \text{CDS}, X) \) is

\[
P(\ln V, \Theta_Y | S, \text{CDS}, X) \\
\propto P_{\text{CDS},S}(\text{CDS}, S | \ln V, \Theta_Y, X) P_Y(V | X, \Theta_Y) P_{\Theta}(\Theta_Y)
\]

\[
= P_S(S | \ln V, \Theta_Y) P_{\text{CDS}}(\text{CDS} | \ln V, \Theta_Y)
\]

\[
\times P_Y(V | X, \Theta_Y) P_{\Theta}(\Theta_Y)
\]

\[
= \prod_{i=1}^{N} \prod_{t=1}^{T} P_S(\ln S_{it} | \ln V_{it}, \Theta_Y) P_{\text{CDS}}(\text{CDS}_{it} | \ln V_{it}, \Theta_Y)
\]

\[
\times P_Y(\ln V_{it} | \ln V_{it-1}, X_{i:t-1}, \Theta_Y) P_{\Theta}(\Theta_Y),
\]

where

\[
P_Y(\ln V_{it} | \ln V_{it-1}, X_{i:t-1}, \Theta_Y) \sim N(\mu_{V_{it}} | V_{it-1}, \sigma_{V_{it}} | V_{it-1}),
\]

\[
P_S(\ln S_{it} | \ln V_{it}, \Theta_Y) \sim N(g_S(V_{it}, \sigma_V, T_D, D), \sigma_S),
\]

\[
P_{\text{CDS}}(\ln \text{CDS}_{it} | \ln V_{it}, \Theta_Y) \sim N(g_{\text{CDS}}(V_{it}, \sigma_{V}, T_{\text{CDS}}, \sigma_{\text{CDS}}).
\]

Thus, stock and CDS markets are assumed to be conditionally independent given asset values. \( \mu_{V_{it}} | V_{it-1}, \sigma_{V_{it}} | V_{it-1} \) are described in (2.8) and (2.9). The \( g_S(\cdot) \) and \( g_{\text{CDS}}(\cdot) \) functions are the equity and CDS pricing solution under the Black–Cox model and are given in online Appendixes A and B, respectively.

For the \( \mu_i \) and \( \sigma_i \), noninformative priors are assumed. For \( \mu_{a1}, \sigma_{a1}, \mu_{a2}, \sigma_{a2}, \mu_{a3} \) and \( \sigma_{a3} \), a flat (uniform) prior within the reasonably bounded interval is assumed, because of the proper posterior distribution. For the prior of hyper-mean parameters, \( \mu_{a1}, \mu_{a2} \) and \( \mu_{a3} \), uniform distribution on \((-1000, 1000)\) interval is assumed. For the prior of hyper-standard deviation parameters, \( \sigma_{a1}, \sigma_{a2} \) and \( \sigma_{a3} \), uniform distribution on \((0, 100)\) is assumed. Because \( a_{1i}, a_{2i} \) and \( a_{3i} \) are constraints in the \((-1, 1)\) interval, the mean and standard deviation that hyper-priors are defined on beyond these intervals are not reasonable.

To obtain Markov chain Monte Carlo samples from the joint posterior distribution, the conditional sampling steps are iterated with the following target distributions:

- \( \ln V_{it} \) is sampled from

\[
P[\ln V_{it} | \Theta_Y, X_{1:1:t}, X_{2:1:t}, X_{3:1:t}, S_{1:1:t}, \text{CDS}_{1:1:t}]
\]

\[
\propto P_Y(\ln V_{it} | \ln V_{it-1}, X_{1:t-1}, X_{2,t-1}, X_{3:t-1}, \Theta_Y)
\]

\[
\times P_Y(\ln V_{it+1} | \ln V_{it}, X_{1:t-1}, X_{2,t-1}, X_{3:t-1}, \Theta_Y)
\]

\[
\times P_S(\ln S_{it} | \ln V_{it}, \Theta) P_{\text{CDS}}(\ln \text{CDS}_{it} | \ln V_{it}, \Theta).
\]
\[ P[\mu_i \mid \ln V_{i,1:T}, \Theta V_{[-\mu_i]}, X_{1,1:T}, X_{2,1:T}, X_{3,1:T}, S_{i,1:T}, \text{CDS}_{i,1:T}] \]
\[ \sim N \left( \frac{T \ln V_{it}^*}{\sigma_i^2(1 - a_{1i}^2 - a_{2i}^2 - a_{3i}^2)/\Delta t}, \left( \frac{T}{\sigma_i^2(1 - a_{1i}^2 - a_{2i}^2 - a_{3i}^2)/\Delta t} \right)^{-1} \right), \tag{3.6} \]

where
\[ \ln V_{it}^* = \ln V_{it} - \ln V_{i,t-1} - \sigma_i (a_{1i} \Delta X_{1t} + a_{2i} \Delta X_{2t} + a_{3i} \Delta X_{3t}) + \frac{\sigma_i^2}{2} \]

and
\[ \overline{\ln V_{it}^*} = \frac{1}{T} \sum_{t=1}^{T} \ln V_{it}^*. \]

* \((a_{1i}, a_{2i}, a_{3i})\) is sampled from
\[ P[a_{1i}, a_{2i}, a_{3i} \mid \ln V_{i,1:T}, \Theta V_{[(a_{1i}, a_{2i}, a_{3i})]}, X_{1,1:T}, X_{2,1:T}, X_{3,1:T}, S_{i,1:T}, \text{CDS}_{i,1:T}] \]
\[ \propto \prod_{t=1}^{T} P_V(\ln V_{it} \mid \ln V_{i,t-1}, X_{1,t-1:T}, X_{2,t-1:T}, X_{3,t-1:T}, \Theta V) \times P_{a1}(a_{1i} \mid \mu_{a1}, \sigma_{a1}) \times P_{a2}(a_{2i} \mid \mu_{a2}, \sigma_{a2}) P_{a3}(a_{3i} \mid \mu_{a3}, \sigma_{a3}) P_{ma1}(\mu_{a1}, \sigma_{a1}) \times P_{ma2}(\mu_{a2}, \sigma_{a2}) P_{ma3}(\mu_{a3}, \sigma_{a3}). \tag{3.7} \]

* \(\sigma_i\) is sampled from
\[ P[\sigma_{V_i} \mid \ln V_{i,1:T}, \Theta V_{[-\sigma_i]}, X_{1,1:T}, X_{2,1:T}, X_{3,1:T}, S_{i,1:T}, \text{CDS}_{i,1:T}] \]
\[ \propto \prod_{t=1}^{T} P_V(\ln V_{it} \mid \ln V_{i,t-1}, X_{1,t-1:T}, X_{2,t-1:T}, X_{3,t-1:T}, \Theta V) \times P_S(\ln S_{it} \mid \ln V_{it}, \Theta V) P_{\text{CDS}}(\ln \text{CDS}_{it} \mid \ln V_{it}, \Theta V) P_\sigma(\sigma_i). \tag{3.8} \]

Hyperparameters \((\mu_{a1}, \sigma_{a1}), (\mu_{a2}, \sigma_{a2})\) and \((\mu_{a3}, \sigma_{a3})\) are sampled with the MH scheme with the following target distributions:
\[ \mu_{ak}, \sigma_{ak} \sim [\mu_{ak}, \sigma_{ak} \mid a_{k1}, \ldots, a_{kN}] \propto \prod_{i=1}^{N} P_{ak}(a_{ki} \mid \mu_{ak}, \sigma_{ak}) P_{mak}(\mu_{ak}, \sigma_{ak}), \tag{3.9} \]

where \(k = 1, 2, 3\).
As proposal distributions in implementing the MH algorithm, the following distributions are used (in the \( j \)th iteration).

- For \( \ln V_{it} \),
  \[
p(\ln V_{it}^{(j)} | \ln V_{it}^{(j-1)}) \sim \text{Gamma} \left( c_0, \frac{\ln V_{it}^{(j-1)} - \ln(L_{0i} \exp(-\gamma(T_D - t)))}{c_0} + \ln(L_{0i} \exp(-\gamma(T_D - t))) \right).
  \]

- For \( \sigma_{V_i} \),
  \[
p(\sigma_{V_i}^{(j)} | \sigma_{V_i}^{(j-1)}) \sim \text{Gamma} \left( c_1, \frac{\sigma_{V_i}^{(j-1)}}{c_0} \right).
  \]

- For \( a_{ki} \), where \( k = 1, 2, 3 \),
  \[
p(a_{ki}^{(j)} | a_{1i}^{(j-1)}, a_{2i}^{(j-1)}, a_{3i}^{(j-1)}) \sim I_{(a_{1i}^{(j)})^2 + (a_{2i}^{(j-1)})^2 + (a_{3i}^{(j-1)})^2 \leq 1} \times \text{Uniform} \left( \max(a_{ki}^{(j-1)} - c_2, -1), \min(a_{ki}^{(j-1)} + c_2, 1) \right).
  \]

\( c_0 \), \( c_1 \) and \( c_2 \) are tuning parameters controlling the step size. Instead of sampling \( \ln V_{it} \) directly, \( \ln V_{it} - \ln(L_{0i} \exp(-\gamma(T_D - t))) \) is sampled from gamma distribution. Because default has not yet occurred, \( \ln V_{it} \) is lower bounded with boundary value \( \ln(L_{0i} \exp(-\gamma(T_D - t))) \) under the Black–Cox model.

The problem that occurs in implementing Gibbs and MH algorithms is posterior chain convergence. Because there are many unknowns to be estimated and the correlation between them is high, it is difficult to achieve convergence of posterior samples. Furthermore, updating one parameter at a time not only is computationally inefficient but can also lead to its being trapped near local extreme values. For example, the posterior distribution of \( \ln V_{i,1:T} \) is heavily dependent on \( \sigma_{V_j} \). If the sampled \( \sigma_j \) values are small, then the volatility of the sampled \( \ln V_{i,1:T} \) again becomes small because the posterior sampling of \( \ln V_{i,1:T} \) is based on the previously sampled \( \sigma_{V_j} \). Once the chain becomes trapped in this circle, it is hard for it to get back out when it updates one parameter at a time. To solve this problem, Liu and Sabatti (2000) and Kou et al (2005) adjusted a simple Gibbs sampling method by adopting the idea of simultaneous updates of correlated unknowns. By modifying the generalized Gibbs sampler, in this paper we propose the following selectively simultaneous move of asset process \( \ln V_{i,1:T} \) and its volatility \( \sigma_{V_j} \):

\[
((\ln V_{i,1:T} - \ln V_{i,1}), \sigma_i) \rightarrow (s'_\nu(\ln V_{i,1:T} - \ln V_{i,1}), s_\nu\sigma_i), \quad (3.10)
\]
where \( s_\sigma \) is a scalar and \( s'_v = (s_v(1), \ldots, s_v(j), \ldots, s_v(T)) \) is a \( 1 \times T \) vector. Each element of \( s'_v \) is determined as follows:

\[
s'_v(j) = \begin{cases} 
  s_\sigma & \text{if } \frac{P(s_\sigma (\ln V_{it} - \ln V_1) \mid s_\sigma \sigma_i X, S, \text{CDS}, \mu)}{P((\ln V_{it} - \ln V_1) \mid \sigma_i, X, S, \text{CDS}, \mu)} > 1, \\
  1 & \text{otherwise,}
\end{cases}
\]  

(3.11)

where \( P((\ln V_{it} - \ln V_1) \mid \sigma_i, X, S, \text{CDS}, \mu) \) is calculated from the conditional posterior distribution of \( \ln V_{it} \), \( P[\ln V_{it} \mid \Theta, X_1, X_2, X_3, S_i, \text{CDS}_i] \), which is given in (3.5).

Instead of implementing a jump in the entire asset process, \( \ln V_{i,1:T} \), together with \( \sigma_i \), a subset of \( \ln V_{i,1:T} \) whose conditional posterior density increases after jumping is selected. When the subset of \( \ln V_{i,1:T} \) is set to jump simultaneously, the acceptance rate increases without a reduction in the jump size, causing the chain to converge faster.

4 SIMULATION

In this section, we validate the parameter estimation process based on the Bayesian estimation method using hypothetical data. The need to incorporate both equity prices and CDSs is also tested. A one-year daily data set for ten firms is generated. The size of the posterior sample, drawn from the posterior distributions described in Section 3, is 30,000. The burn-in period is 10,000.

The daily asset value processes of ten firms are generated. Based on (2.1) and (2.2), with one observed common factor, we generate a one-year daily asset value process for each firm. Equity and five-year CDS prices are derived from simulated asset value paths based on (2.3) and (2.4).

Similarly to Duffie et al. (2007), with \( V_{it}^0 = (V_{i0}/S_{i0})S_{it} \) and given \( V_{i0} \), we iteratively update \( \mu_i \), \( \sigma_i \) and \( V_{it} \) as follows.

- For \( \mu_i^{(m)} \):

\[
\mu_i^{(m)} = \frac{\sum_{t=1}^{T}(V_{it}^{(m)} - V_{i,t-1}^{(m)})}{T - 1}.
\]

- For \( \sigma_i^{(m)} \):

\[
\sigma_i^{(m)} = \left( \frac{\sum_{t=1}^{T}(V_{it}^{(m)} - V_{i,t-1}^{(m)} - \mu_i^{(m)})^2}{T - 1} \right)^{1/2}.
\]

- For \( V_{it}^{(m)} \):

\[
\arg\min_{V_{it}^{(m)}} (g_S(V_{it}^{(m)}, \sigma_i^{(m)}, T_D, D) - S_{it})^2.
\]
TABLE 1

<table>
<thead>
<tr>
<th>Firm</th>
<th>True $\alpha_i$</th>
<th>$\beta_i$</th>
<th>$\gamma_i$</th>
<th>$\delta_i$</th>
<th>$\epsilon_i$</th>
<th>$\phi_i$</th>
<th>$\psi_i$</th>
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<tr>
<td>2</td>
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<td>0.459</td>
<td>0.390</td>
<td>0.230</td>
<td>0.472</td>
<td>0.100</td>
<td>0.532</td>
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<tr>
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<td>0.002</td>
<td>0.001</td>
<td>0.003</td>
<td>0.005</td>
<td>0.003</td>
<td>0.000</td>
<td>0.001</td>
<td>0.003</td>
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<tr>
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</tr>
</tbody>
</table>

This table summarizes the posterior mean and posterior standard deviation of all ten simulated asset volatilities and common factor coefficients. SD denotes standard deviation.
FIGURE 1 ACFs of posterior sample of volatility $\sigma_i$ in the simulation study.

(a) Only equity prices (stock market information) used. (b) CDSs and equity both used.

We continue with this iteration until all of the $V_{it}$, $\mu_i$ and $\sigma_i$ converge, and then set these converged values as the initial configurations of their posterior samples. The posterior samples of all other parameters (hyperparameters and random coefficients) are initialized with random points. For the default barrier function in equation (A.2) in the online appendix, $\gamma = 0.02$ and $T_D = 10$ years are assumed. The recovery rate is set to 0.51, the assumed median recovery rate in the historical data, for all firms.
In order to determine not only the accuracy of the proposed Bayesian method but also the improvement in the estimation performance from the additional use of CDS spreads, the model is estimated for two cases: using only stock prices and using both equity and CDS price information. All of the posterior distributions of the parameters correctly capture their true values, with similar variances in both cases. All asset value paths are also correctly estimated in both cases.

However, after adding the CDS information, significant improvements are seen in the estimation of the volatility parameters $\sigma_i$. Table 1 summarizes the simulated posterior distribution. The posterior mean is closer to the true values, and the posterior standard deviation is much smaller when both CDS and stock market information is used. In terms of both estimation accuracy and efficiency, adding in CDS spread information provides significant improvements.

Furthermore, their posterior chain converges more quickly when the CDS information is added. Figure 1(b) shows the autocorrelation functions (ACFs) of the posterior sample of volatility $\sigma_i$ when both CDS and stock information is used in the estimation. Figure 1(a) shows the ACFs of the posterior sample of volatility $\sigma_i$ when only stock information is used in the estimation.

5 EMPIRICAL STUDY

Financial data on firms constituting the CDX.NA.IG13 was collected for the empirical study. Daily spreads of the 125 issuers’ five-year CDSs constituting the CDX.NA.IG13 were obtained from Bloomberg. The data covers the period from January 2, 2009 to December 31, 2009 and consists of mid-quote spreads. The empirical study is based on one-year data in order to easily apply the Bayesian method and make practical use of the estimated results. Hu et al (2013) pointed out that credit risk models are typically estimated over a one-year horizon. Because the year 2009 contains both volatile and stable paths of stock prices, more diverse sample paths can be obtained in that year than in other periods.

Of the 125 issuers, we use only the 118 issuers with no missing historical stock price data and no missing information on their balance sheets. The seven issuers omitted were AT&T Mobility, COX Communications, International Lease Finance, Motorola Solutions, National Rural Utility Coop, News Corporation and Yum! Brands. Stock information was obtained from the Center for Research in Security Prices (CRSP), and balance sheet information from Compustat. For the fixed interest rate, we used the average of the one-year US Treasury constant maturity rate obtained from the Federal Reserve Bank, which was 0.0047 during 2009.

The sum of the liability and equity values in December 2008 was set as the first value of the assets of each firm, $V_{i0}$. The posterior samples were drawn 10000 times
with a 5000 burn-in period. The SMB effect, $a_{3i}$, was eventually excluded, because its effect was not large enough for us to estimate.

5.1 Calibration of default barrier and recovery rate

An additional parameter, the recovery rate $r$, is needed in order for us to incorporate the CDS price into the model. However, it is well understood that the unknown default barrier parameters, asset values and recovery rate under the structural model are not simultaneously identifiable (Das and Hanouna 2009). For this reason, most papers assume the recovery rate to be an exogenously supplied value. Using the iterative scheme proposed by Vassalou and Xing (2004), Hull et al (2010) found default barrier parameters under a fixed recovery rate. They used the CDX.NA.IG index level at the mid-date of the data period and derived common default barrier parameters for all firms.

Instead of a common default barrier parameter, we set an individual $L_{0i}$ level in the default barrier function in (A.2) given in the online appendix. Similarly to Duan and Fulop (2009) and Kwon and Lee (2016), using the December 31, 2008 balance sheet data from Compustat, we calculate the long-term-debt-to-common-equity ratio. Then, multiplying by the observed equity value (the product of the stock price and number of shares outstanding), we have the $L_{0i}$ level. Another default barrier parameter $\gamma$ in (A.2) is set to 0.02, as in Lando (2004).2 The recovery rate is fixed at 0.51.3

5.2 Validation of CDS spreads in the estimation

In order to validate the model with additional financial market information, we fitted it first using equity prices only and then using both equity and CDS prices. The empirical results show that both equity prices and CDSs are necessary in the model estimation. In terms of the posterior chain convergence and efficiency of the estimated values, using both equity prices and CDS market information improved the performance of the estimation in comparison with just using equity prices.

Similarly to what we described in terms of the simulation results in Section 4, algorithms converge faster in cases using CDS data. Parts (a) and (b) of Figure 2 show the ACF of the posterior sample of hyperparameters, that is, the mean and standard deviation of the random coefficients of SP500 and HML. The convergence

---

2 Previous research on structural credit risk models usually assume $\gamma$ takes a value between 0 and 0.04. We found that, within this range, choosing a fixed value for $\gamma$ did not significantly affect the determination of $L_{0i}$.

3 We note that, regardless of using a recovery rate of 0.3 (the assumed mezzanine tranche recovery rate in Jobst et al (2015)) or 0.2 (the assumed equity tranche recovery rate in Jobst et al (2015)), the parameter estimates proved to be stable.
The ACF of the posterior sample of hyperparameters is shown in Figure 2. The ACFs are calculated for two different scenarios: (a) only equity prices used and (b) CDSs and equity both used. The figures show that the use of CDS information leads to a significant improvement in the estimation of the hyperparameters.

Rates are all greatly improved in the hyperparameter sampling. A similar improvement is observed in the sampling of the individual parameters $a_{1i}$, $a_{2i}$, and $\sigma_i$.

Another benefit of adding CDS information is greater estimation efficiency. Figure 3 shows a box plot of the change in posterior standard deviation after adding CDSs. Except for the mean of the asset process (marked $\mu_i$), the posterior sample based on both CDS and stock information has a smaller posterior standard deviation. This is because equity and CDS prices are not incorporated into the estimation of $\mu_i$, as shown in (3.6). The asset price volatility estimation improves, especially in terms of estimation efficiency, with the addition of CDS information.

Because the estimation performance is improved by the addition of CDSs, the results provided in the rest of the paper are based on the use of both CDS and equity prices for the estimation.

### 5.3 Results

We summarize the estimated results based on the posterior distribution. Table 2 shows the posterior mean and posterior standard deviation of parameters $a_{1i}$, $a_{2i}$, $\mu_i$ and $\sigma_i$ for each firm in (2.1) and (2.2). Random coefficient $a_{1i}$ represents the effect of
FIGURE 3  Box plots of difference in posterior standard deviation with and without the use of CDS prices.

This graph shows the change in the posterior standard deviation, before and after the addition of CDS prices, in estimating (from the left) the SP500 coefficient ($a_{1i}$), the HML coefficient ($a_{2i}$), the mean level of the asset process ($\mu_i$) and the volatility level ($\sigma_i$) of the asset process. The four box plots show the posterior standard deviation when using only $S$ minus the posterior standard deviation when using both CDS and $S$.

the market return (S&P 500) process on firm $i$’s asset path. For the firm coded “L”, the estimated value of $a_{1i}$ is 0.873. On the other hand, for firm coded “CTL”, the estimated value of $a_{1i}$ is only 0.257. The estimated value of $a_{2i}$ represents the effect of the value premium (HML) on firm $i$’s asset path. In most firms, the sign of the coefficient of HML is negative, and this unexpected negative sign occurs because of the correlation between the factors.

Under the structural correlated model proposed in this paper, we can calculate the asset paths’ correlation with the cross-product of random coefficients as in (2.10). Hence, the relative sizes of the estimated common factor coefficients $a_{1i}$ and $a_{2i}$ allow us to infer the relative sizes of the correlation coefficients between asset paths. The estimated values of $a_{1i}$ and $a_{2i}$ for firm CTL are relatively smaller than those for the other firms. Thus, the asset path of CTL is expected to be less correlated with the other firms’ asset paths.

Table 3 shows that the coefficients of SP500 are more concentrated, and larger than the HML coefficients. The overall magnitude of the common factor effect and the degree of soft constraint or soft homogeneity applied to the random coefficients $a_{1i}$ and $a_{2i}$ can be compared using the estimated hyperparameters. Table 3 shows the posterior mean and posterior standard deviation of the hyperparameters $\mu_{a_1}$, $\sigma_{a_1}$, $\mu_{a_2}$ and $\sigma_{a_2}$ in (2.5).
### TABLE 2  Firm-by-firm parameters. [Table continues on next two pages.]

<table>
<thead>
<tr>
<th>Ticker</th>
<th>$a_{1i}$</th>
<th>$a_{2i}$</th>
<th>$\mu_i$</th>
<th>$\sigma_i$</th>
</tr>
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<tbody>
<tr>
<td>AA</td>
<td>0.774 (0.019)</td>
<td>-0.224 (0.036)</td>
<td>0.481 (0.359)</td>
<td>0.597 (0.004)</td>
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<tr>
<td>ABX</td>
<td>0.566 (0.030)</td>
<td>-0.391 (0.041)</td>
<td>0.362 (0.464)</td>
<td>0.641 (0.002)</td>
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<tr>
<td>ACE</td>
<td>0.782 (0.018)</td>
<td>-0.363 (0.031)</td>
<td>0.169 (0.336)</td>
<td>0.660 (0.002)</td>
</tr>
<tr>
<td>AEP</td>
<td>0.661 (0.025)</td>
<td>-0.389 (0.036)</td>
<td>0.028 (0.148)</td>
<td>0.233 (0.001)</td>
</tr>
<tr>
<td>AET</td>
<td>0.699 (0.024)</td>
<td>-0.319 (0.037)</td>
<td>0.109 (0.305)</td>
<td>0.483 (0.002)</td>
</tr>
<tr>
<td>AIG</td>
<td>0.532 (0.041)</td>
<td>-0.087 (0.071)</td>
<td>0.019 (0.357)</td>
<td>0.430 (0.007)</td>
</tr>
<tr>
<td>ALL</td>
<td>0.803 (0.021)</td>
<td>-0.090 (0.044)</td>
<td>0.017 (0.280)</td>
<td>0.472 (0.002)</td>
</tr>
<tr>
<td>AMGN</td>
<td>0.696 (0.020)</td>
<td>-0.539 (0.024)</td>
<td>0.017 (0.211)</td>
<td>0.448 (0.002)</td>
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<tr>
<td>APC</td>
<td>0.807 (0.016)</td>
<td>-0.311 (0.030)</td>
<td>0.484 (0.271)</td>
<td>0.543 (0.002)</td>
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<td>ARW</td>
<td>0.806 (0.016)</td>
<td>-0.369 (0.026)</td>
<td>0.428 (0.258)</td>
<td>0.556 (0.002)</td>
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<tr>
<td>AVT</td>
<td>0.777 (0.016)</td>
<td>-0.415 (0.025)</td>
<td>0.642 (0.341)</td>
<td>0.745 (0.003)</td>
</tr>
<tr>
<td>AXP</td>
<td>0.653 (0.041)</td>
<td>0.120 (0.069)</td>
<td>0.156 (0.159)</td>
<td>0.213 (0.002)</td>
</tr>
<tr>
<td>AZO</td>
<td>0.535 (0.029)</td>
<td>-0.431 (0.037)</td>
<td>-0.098 (0.158)</td>
<td>0.215 (0.002)</td>
</tr>
<tr>
<td>BA</td>
<td>0.361 (0.040)</td>
<td>0.029 (0.048)</td>
<td>-0.094 (0.134)</td>
<td>0.145 (0.001)</td>
</tr>
<tr>
<td>BAX</td>
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<td>0.055 (0.198)</td>
<td>0.381 (0.001)</td>
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<tr>
<td>BDK</td>
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<td>0.185 (0.211)</td>
<td>0.294 (0.002)</td>
</tr>
<tr>
<td>BMY</td>
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<td>0.382 (0.001)</td>
</tr>
<tr>
<td>BNI</td>
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<td>0.337 (0.001)</td>
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<tr>
<td>BXP</td>
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<td>-0.050 (0.045)</td>
<td>0.134 (0.198)</td>
<td>0.326 (0.002)</td>
</tr>
<tr>
<td>CAG</td>
<td>0.716 (0.019)</td>
<td>-0.524 (0.023)</td>
<td>0.226 (0.181)</td>
<td>0.401 (0.001)</td>
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<tr>
<td>CAH</td>
<td>0.688 (0.025)</td>
<td>-0.371 (0.036)</td>
<td>0.016 (0.273)</td>
<td>0.432 (0.002)</td>
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<tr>
<td>CAT</td>
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<td>-0.069 (0.131)</td>
<td>0.141 (0.001)</td>
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<tr>
<td>CB</td>
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<td>0.068 (0.261)</td>
<td>0.528 (0.002)</td>
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<tr>
<td>CBS</td>
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<td>-0.043 (0.050)</td>
<td>0.380 (0.322)</td>
<td>0.492 (0.003)</td>
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<tr>
<td>CCL</td>
<td>0.819 (0.015)</td>
<td>-0.285 (0.030)</td>
<td>0.352 (0.313)</td>
<td>0.627 (0.003)</td>
</tr>
<tr>
<td>CEG</td>
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<td>-0.437 (0.029)</td>
<td>0.132 (0.182)</td>
<td>0.326 (0.002)</td>
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<tr>
<td>CI</td>
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<td>0.655 (0.347)</td>
<td>0.607 (0.003)</td>
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<tr>
<td>CMCSA</td>
<td>0.786 (0.018)</td>
<td>-0.309 (0.031)</td>
<td>0.007 (0.208)</td>
<td>0.391 (0.002)</td>
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<tr>
<td>CNQ</td>
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<td>0.488 (0.282)</td>
<td>0.552 (0.003)</td>
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<td>COF</td>
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<td>0.228 (0.312)</td>
<td>0.426 (0.002)</td>
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<td>-0.017 (0.172)</td>
<td>0.374 (0.001)</td>
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<td>CPB</td>
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<td>-0.033 (0.131)</td>
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<td>CSC</td>
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<td>CSCEO</td>
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<td>-0.468 (0.019)</td>
<td>0.526 (0.268)</td>
<td>0.723 (0.002)</td>
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<td>CSX</td>
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<td>-0.230 (0.034)</td>
<td>0.206 (0.186)</td>
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<td>CTL</td>
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<td>0.515 (0.002)</td>
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<td>CVS</td>
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<td>Ticker</td>
<td>$a_{1i}$</td>
<td>$a_{2i}$</td>
<td>$\mu_i$</td>
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<td>DE</td>
<td>0.716 (0.028)</td>
<td>-0.022 (0.050)</td>
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<td>DELL</td>
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<td>0.238 (0.002)</td>
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<td>DUK</td>
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<td>0.383 (0.001)</td>
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<td>DYN</td>
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<td>EQR</td>
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<td>0.009 (0.168)</td>
<td>0.229 (0.002)</td>
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<td>FE</td>
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<td>-0.049 (0.158)</td>
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<td>FO</td>
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<td>0.112 (0.178)</td>
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<td>GMT</td>
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<td>-0.110 (0.142)</td>
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<td>0.656 (0.002)</td>
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<td>IBM</td>
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<td>0.114 (0.130)</td>
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<td>0.285 (0.002)</td>
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<td>IR</td>
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<td>0.666 (0.318)</td>
<td>0.600 (0.002)</td>
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<tr>
<td>JCI</td>
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<td>-0.321 (0.031)</td>
<td>0.657 (0.444)</td>
<td>0.761 (0.004)</td>
</tr>
<tr>
<td>JWN</td>
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<td>0.476 (0.223)</td>
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<td>KFT</td>
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<td>0.009 (0.167)</td>
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<td>-0.517 (0.025)</td>
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<td>KR</td>
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<td>-0.159 (0.148)</td>
<td>0.231 (0.001)</td>
</tr>
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<td>KSS</td>
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<td>0.506 (0.308)</td>
<td>0.668 (0.002)</td>
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<td>-0.132 (0.159)</td>
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<td>0.157 (0.262)</td>
<td>0.541 (0.002)</td>
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<td>LUV</td>
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<td>-0.286 (0.043)</td>
<td>0.229 (0.316)</td>
<td>0.463 (0.002)</td>
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<td>MAR</td>
<td>0.766 (0.020)</td>
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<td>0.082 (0.155)</td>
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<tr>
<td>MCD</td>
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<td>-0.042 (0.154)</td>
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<td>MET</td>
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<td>0.076 (0.051)</td>
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<td>0.530 (0.003)</td>
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</table>
TABLE 2  Continued.

<table>
<thead>
<tr>
<th>Ticker</th>
<th>$a_{1i}$</th>
<th>$a_{2i}$</th>
<th>$\mu_i$</th>
<th>$\sigma_i$</th>
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<tr>
<td>MMC</td>
<td>0.734 (0.021)</td>
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<td>-0.024 (0.222)</td>
<td>0.390 (0.001)</td>
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<td>-0.002 (0.137)</td>
<td>0.191 (0.001)</td>
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<td>0.752 (0.017)</td>
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<td>0.257 (0.241)</td>
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<td>NSC</td>
<td>0.807 (0.019)</td>
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<td>0.048 (0.189)</td>
<td>0.356 (0.001)</td>
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<td>NWL</td>
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<td>-0.036 (0.161)</td>
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<td>0.263 (0.251)</td>
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<td>RIG</td>
<td>0.819 (0.014)</td>
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<td>0.715 (0.019)</td>
<td>-0.512 (0.024)</td>
<td>0.059 (0.261)</td>
<td>0.550 (0.002)</td>
</tr>
<tr>
<td>SHW</td>
<td>0.741 (0.018)</td>
<td>-0.465 (0.025)</td>
<td>0.193 (0.318)</td>
<td>0.652 (0.002)</td>
</tr>
<tr>
<td>SLE</td>
<td>0.707 (0.022)</td>
<td>-0.416 (0.031)</td>
<td>0.084 (0.183)</td>
<td>0.319 (0.001)</td>
</tr>
<tr>
<td>SPG</td>
<td>0.420 (0.041)</td>
<td>0.153 (0.054)</td>
<td>-0.008 (0.149)</td>
<td>0.166 (0.001)</td>
</tr>
<tr>
<td>SPLS</td>
<td>0.751 (0.014)</td>
<td>-0.509 (0.019)</td>
<td>0.912 (0.477)</td>
<td>1.133 (0.004)</td>
</tr>
<tr>
<td>SRE</td>
<td>0.760 (0.017)</td>
<td>-0.467 (0.024)</td>
<td>0.183 (0.170)</td>
<td>0.382 (0.002)</td>
</tr>
<tr>
<td>SWY</td>
<td>0.711 (0.020)</td>
<td>-0.479 (0.027)</td>
<td>-0.076 (0.181)</td>
<td>0.357 (0.002)</td>
</tr>
<tr>
<td>T</td>
<td>0.745 (0.018)</td>
<td>-0.469 (0.026)</td>
<td>0.016 (0.191)</td>
<td>0.402 (0.002)</td>
</tr>
<tr>
<td>TGT</td>
<td>0.785 (0.018)</td>
<td>-0.389 (0.027)</td>
<td>0.201 (0.176)</td>
<td>0.359 (0.002)</td>
</tr>
<tr>
<td>TJX</td>
<td>0.742 (0.017)</td>
<td>-0.490 (0.022)</td>
<td>0.578 (0.277)</td>
<td>0.604 (0.002)</td>
</tr>
<tr>
<td>TOL</td>
<td>0.769 (0.021)</td>
<td>-0.204 (0.038)</td>
<td>-0.009 (0.263)</td>
<td>0.435 (0.002)</td>
</tr>
<tr>
<td>TWC</td>
<td>0.361 (0.040)</td>
<td>0.005 (0.050)</td>
<td>-0.166 (0.266)</td>
<td>0.291 (0.002)</td>
</tr>
<tr>
<td>TWX</td>
<td>0.739 (0.027)</td>
<td>-0.112 (0.049)</td>
<td>-0.069 (0.201)</td>
<td>0.307 (0.001)</td>
</tr>
<tr>
<td>UHS</td>
<td>0.736 (0.019)</td>
<td>-0.461 (0.026)</td>
<td>0.374 (0.238)</td>
<td>0.477 (0.002)</td>
</tr>
<tr>
<td>UNH</td>
<td>0.696 (0.020)</td>
<td>-0.448 (0.027)</td>
<td>0.156 (0.293)</td>
<td>0.529 (0.003)</td>
</tr>
<tr>
<td>UNP</td>
<td>0.824 (0.016)</td>
<td>-0.330 (0.027)</td>
<td>0.227 (0.201)</td>
<td>0.444 (0.002)</td>
</tr>
<tr>
<td>UPS</td>
<td>0.774 (0.019)</td>
<td>-0.336 (0.031)</td>
<td>-0.007 (0.138)</td>
<td>0.258 (0.001)</td>
</tr>
<tr>
<td>VIA</td>
<td>0.814 (0.015)</td>
<td>-0.335 (0.026)</td>
<td>0.241 (0.174)</td>
<td>0.367 (0.002)</td>
</tr>
<tr>
<td>VLO</td>
<td>0.748 (0.019)</td>
<td>-0.341 (0.033)</td>
<td>0.003 (0.359)</td>
<td>0.621 (0.003)</td>
</tr>
<tr>
<td>VNO</td>
<td>0.684 (0.038)</td>
<td>0.143 (0.065)</td>
<td>0.044 (0.159)</td>
<td>0.223 (0.002)</td>
</tr>
<tr>
<td>VZ</td>
<td>0.695 (0.022)</td>
<td>-0.409 (0.032)</td>
<td>-0.057 (0.160)</td>
<td>0.265 (0.001)</td>
</tr>
<tr>
<td>WFC</td>
<td>0.372 (0.031)</td>
<td>0.355 (0.035)</td>
<td>-0.068 (0.125)</td>
<td>0.150 (0.001)</td>
</tr>
<tr>
<td>WHR</td>
<td>0.805 (0.017)</td>
<td>-0.235 (0.035)</td>
<td>0.575 (0.315)</td>
<td>0.580 (0.003)</td>
</tr>
<tr>
<td>WMT</td>
<td>0.680 (0.020)</td>
<td>-0.557 (0.022)</td>
<td>-0.005 (0.197)</td>
<td>0.411 (0.001)</td>
</tr>
<tr>
<td>XL</td>
<td>0.765 (0.023)</td>
<td>-0.025 (0.048)</td>
<td>1.725 (0.632)</td>
<td>0.965 (0.004)</td>
</tr>
<tr>
<td>XRX</td>
<td>0.773 (0.021)</td>
<td>-0.269 (0.034)</td>
<td>0.018 (0.200)</td>
<td>0.348 (0.002)</td>
</tr>
<tr>
<td>XTO</td>
<td>0.778 (0.019)</td>
<td>-0.312 (0.032)</td>
<td>0.222 (0.243)</td>
<td>0.451 (0.002)</td>
</tr>
</tbody>
</table>

This table summarizes the posterior mean (posterior standard deviation) of all parameters.
This figure shows all estimated $a_{1i}$ and their 95% posterior interval (PI). The dotted lines refer to the 95% PIs of companies with a relatively small market return effect (bottom 25%). The dashed lines refer to the 95% PIs of companies with a relatively large market return effect (top 25%). The rest of the companies’ 95% PIs are shown by solid lines.

**TABLE 3** Estimated hyperparameters.

<table>
<thead>
<tr>
<th></th>
<th>$\mu_{a_1}$</th>
<th>$\mu_{a_2}$</th>
<th>$\sigma_{a_1}$</th>
<th>$\sigma_{a_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Posterior mean</td>
<td>0.715</td>
<td>-0.300</td>
<td>0.080</td>
<td>0.148</td>
</tr>
<tr>
<td>Posterior standard deviation</td>
<td>0.007</td>
<td>0.014</td>
<td>0.004</td>
<td>0.008</td>
</tr>
</tbody>
</table>

This table summarizes the posterior mean and posterior standard deviation of all hyperparameters.

The overall magnitude of the effect of SP500 is bigger than that of HML; the hyper-mean of $a_1$, $\mu_{a_1}$, is estimated as 0.716, but the hyper-mean of $a_2$, $\mu_{a_2}$, is estimated as $-0.299$. However, the effects of SP500 are more homogeneous across the firms than the effects of HML; the hyper-standard-deviation of $a_1$, $\sigma_{a_1}$, is estimated as 0.080, but the hyper-standard-deviation of $a_2$, $\sigma_{a_2}$, is estimated as 0.148.

Based on the above empirical results, we can validate that the random coefficients assumption improves the model. Figure 4 shows the posterior mean and 95% posterior interval (PI) of the SP500 coefficients $a_{1i}$. As shown in Figure 4 and Table 2, there is significant variation in $a_{1i}$ and even more significant variation in $a_{2i}$. This suggests that the approach in Duffie et al (2009) of letting all firms have the same sensitivity to the market is not supported by the data. However, the degree of homogeneity in the effects of SP500 and HML cannot be assumed to be same, as shown in Table 3. By partially pooling (or connecting) the coefficients, the random coefficient model compromises between two extremes: equal factor effects and completely different factor effects across firms.
TABLE 4 Summary of CDX.NA.IG13 quotes (%) between September 23, 2009 and December 31, 2009 (68 days).

<table>
<thead>
<tr>
<th>Tranche</th>
<th>0–3%</th>
<th>3–7%</th>
<th>7–10%</th>
<th>10–15%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min</td>
<td>48</td>
<td>0</td>
<td>5.813</td>
<td>0.5</td>
</tr>
<tr>
<td>1st quantile</td>
<td>51.7</td>
<td>20.47</td>
<td>7.813</td>
<td>1</td>
</tr>
<tr>
<td>Median</td>
<td>55.4</td>
<td>22.12</td>
<td>8.312</td>
<td>1.625</td>
</tr>
<tr>
<td>Mean</td>
<td>55.4</td>
<td>21.39</td>
<td>8.215</td>
<td>1.584</td>
</tr>
<tr>
<td>3rd quantile</td>
<td>58.8</td>
<td>23.12</td>
<td>8.922</td>
<td>2.063</td>
</tr>
<tr>
<td>Max</td>
<td>62.5</td>
<td>26.62</td>
<td>10.690</td>
<td>3.313</td>
</tr>
</tbody>
</table>

6 APPLICATION TO COLLATERALIZED DEBT OBLIGATION VALUATION: MODEL VALIDATION CHECK WITH OUT-OF-SAMPLE DATA

In order to ascertain whether the correlation estimates are in line with the empirical findings, this section introduces the CDX.NA.IG13 tranches. We test how well the models fit market data on the prices of a CDX tranche. Because we do not incorporate CDX tranche price data into the model estimation, the CDX tranche pricing discussed in this section provides a model validation check with out-of-sample data.

Since in order to value a CDX tranche it is necessary to develop a model of the joint probability of default of the issuers in the underlying portfolio, we used the market price of a CDX tranche to test the plausibility of our estimated results. The detailed valuation procedure is discussed in Appendix C online.

The standard tranche structure of CDX.NA.IG is 0–3% (equity tranche), 3–7% (mezzanine tranche), 7–10% and 10–15%. Table 4 shows the summary statistics for five-year maturity daily CDX.NA.IG13 tranche market quotes between September 23, 2009 and December 31, 2009.

The posterior distribution of each CDX tranche price is simulated using the last 5000 posterior draws of the parameters. The simulation method for the CDX tranche prices’ posterior distribution with the $m$th posterior draw of $\ln V_i^{(m)}$, $\sigma_V^{(m)}$, $\mu_i^{(m)}$, $a_{i1}^{(m)}$ and $a_{i2}^{(m)}$ is as follows. The simulation is carried out by drawing a set of daily changes $\Delta \text{SP500}_t$, $\Delta \text{HML}_t$ and $\Delta w_{it}$ for a five-year period. The $\Delta w_{it}$ are sampled independently for every $i$ and $t$ from a normal distribution with mean 0 and variance $\Delta t = 1/252$. $\Delta \text{SP500}_t$ and $\Delta \text{HML}_t$ are sampled from a multivariate normal distribution with a mean 0 vector and a covariance matrix estimated from the real $\Delta \text{SP500}_t$ and $\Delta \text{HML}_t$ during September, 23, 2009 and December 31, 2009. The asset value of
firm $i$ at time $t$ is then derived as follows:

\[
\ln V_{it}^{(m)} = \ln V_{it}^{(m)} + \left( \mu_i^{(m)} - \frac{1}{2\sigma_i^{(m)^2}} \right) \Delta t + \sigma_i^{(m)} (\lambda_i^{SP500} + \lambda_i^{HML}) \sqrt{\Delta t} + \sigma_{\text{Vi}} \sqrt{1 - \lambda_i^{(m)^2} - \lambda_i^{(m)^2} \Delta w_{it}} \sqrt{\Delta t}.
\]

(6.1)

Firm $i$ is assumed to default at the midpoint of the time interval $(t-1, t)$: if the value of $V_{it}^{(m)}$ is below the barrier, then the derivation of $V_i$ stops; if not, then the value $V_{i,t+1}^{(m)}$ is sampled again. This derivation is iterated 100 times for all $i = 1, \ldots, 118$ with each $m$th posterior draw. For $m$, every fiftieth (posterior) sampled $\ln V_{it}^{(m)}, \sigma_{\text{Vi}}^{(m)}, \lambda_i^{1i}$ and $\lambda_i^{2i}$ is used after the burn-in period of 5000. For $\lambda_i^{(m)}$, we use the median value of $\lambda_i^{(m)}$ for all $i = 1, \ldots, 118$. The reason we do not use individual $\lambda_i^{(m)}$ values is that the posterior distribution of $\lambda_i^{(m)}$ for each $i$ is widely spread relative to its mean level, as shown in Table 2. Incorporating such widely distributed values may cause a reduction in prediction efficiency. Lando (2004) pointed out that adding the mean drift of the asset into the model provides almost no information, because it is difficult to estimate the mean drift of the asset accurately. For the same reason, Crossbie (2002) measured the distance to default without consideration of the mean drift of the asset. By using the median value of $(\mu_1^{(m)}, \ldots, \mu_{118}^{(m)})$ as $\mu_i^{(m)}$ for all $i = 1, \ldots, 118$, we reduce the variability of the posterior distribution of $\mu_i$. Hence, among 10,000 sets of posterior-sampled $\ln V_{it}^{(m)}, \sigma_{\text{Vi}}^{(m)}, \text{median}(\mu_1^{(m)}, \ldots, \mu_{118}^{(m)}), \lambda_i^{1i}$ and $\lambda_i^{2i}$, 100 sets of sampled values with $m = 5000, 5050, 5100, \ldots, 10000$ are used.

For each $m$, 100 sets of the 118 issuers’ $V_{i,t}^{\text{m}}$ paths are simulated and denoted by $(V_{i,1:\text{min}(\tau,T)}^{m,1}, \ldots, V_{i,1:\text{min}(\tau,T)}^{m,100})$, where $T$ is the maturity of the CDS and $\tau$ is the default time. For each set of paths, the number of defaults that occur in each time interval $(t-1, t)$ is determined; then, the loss $\text{Loss}_t$ in equation (C.1) of the online appendix can be estimated by $\widehat{\text{Loss}}_t$ as

\[
\widehat{\text{Loss}}_t^{(m)} = \frac{1}{N} \left( \sum_{i=1}^{N} (1 - R_i) \frac{1}{100} \sum_{l=1}^{100} 1_{D_{i,t}^{m,l} = 1} \right),
\]

(6.2)

where $R_i$ is the recovery rate and $D_{i,t}^{m,l}$ is the default indicator of firm $i$ by time $t$ at the $l$th default-time-simulation iteration, with the $m$th posterior draw of parameters. Then,
we get the posterior distribution of the tranche prices by obtaining $\bar{\text{Loss}}_t^m$ for $m = 5000, 5050, 5100, \ldots, 10000$. This posterior distribution simulation of derivatives prices, based on the estimation results of the underlying securities, is one of the advantages gained by applying the Bayesian method in finance.

Figure 5 shows the histograms of the posterior distributions of CDX tranche prices on December 31, 2009. For comparison, the posterior distribution of the CDX quote is also simulated with no correlation structures incorporated. As shown in histograms (e)–(h), when the correlation between the asset processes is not considered the equity tranche (0–3%) is overestimated and the other tranche values are underestimated. However, when we incorporate the common factors, the over- or underestimation problems are resolved.

The results coincide with the findings of Hull and White (2004), who showed the sensitivity of each tranche price to the default correlation. For the equity tranche, a higher correlation means a lower value to someone buying protection. For the mezzanine tranche, the value of the tranche is not particularly sensitive to correlation. For other senior tranches (7% and more), higher correlation means a higher value to someone buying protection.

The problem of overestimation in the equity tranche and underestimation in the 3–7%, 7–10% and 10–15% tranches is resolved by incorporating the SP500 and HML common factors.
7 CONCLUSION

In this paper, we built a structural correlated default model under the Black–Cox setup. By assuming that each asset process depends on common factor paths, SP500 and HML, we modeled the default correlation structure as in Hull et al (2010). Several aspects of the model and estimation process are extended from previous studies. Based on simulation and empirical studies, we validate the need for and advantage of the extensions proposed in our paper.

First, by assuming there is market noise, both stock and CDS price information are used simultaneously for asset process estimation. The simulation and empirical results show that adding CDS prices to the estimation improves posterior sample convergence.

Second, we relax the assumptions of equal factor effects across all firms that have been applied in previous studies. The empirical study based on 2009 stock and CDS price information for 125 firms constituting the CDX.NA.IG13 suggests that Duffie et al’s (2009) hypothesis that all firms have the same sensitivity is not supported by the data. In terms of the overall magnitude of the common effect, SP500 affects firms’ asset paths more than HML. However, in terms of the degree of homogeneity of the common-factor effect across the firms, the SP500 effect is more homogeneous than the HML effect. By assuming random coefficients $a_1i$ and $a_2i$ and estimating the hyperparameters governing them, we can make the coefficients partially pooled or connected, instead of assuming the extreme conditions of complete equality or inequality.

Third, in order to deal with such a complicated model, we adopt Bayesian estimation techniques. By introducing selective simultaneous jumping, we modify the generalized Gibbs sampler given in Liu and Sabatti (2000) and Kou et al (2005). Through a simulation study, we find that chain convergence improves.

The nonequal common-factor-effect model suggests one of the potential uses of this paper. This makes it possible for the pairwise and firm-by-firm correlation structure to be derived. Hence, inferences based on this model can offer a better understanding of specific portfolios’ risk structures, because it is derived from the firm-specific correlation structure.

In order to discuss this potential practical application of our paper, we apply the estimated empirical results to CDX tranche pricing. The posterior distribution of CDX tranche prices can be derived because we adopt the Bayesian estimation method. Incorporating the correlation structure improves the prediction of CDX tranche market quotes. However, unlike in previous CDX tranche pricing studies, in this paper the history of CDX tranche prices is not used for parameter estimation. This means that the prediction of CDX tranche prices also acts as a model validation check with out-of-sample data. Moreover, unlike previous random factor models for CDX tranche
Bayesian estimation of a correlated structural credit risk model with random coefficients

pricing, as in Longstaff and Rajan (2008) and Jobst et al (2015), our correlated credit risk model is derived from the structural credit risk model.

As a model that captures correlated defaults based on a structural credit risk model and an estimation method applicable to deriving the posterior distributions of other financial derivatives, which is not used in the model estimation procedure, the CDX tranche pricing method described in this paper shows a potential application of our approach.

DECLARATION OF INTEREST

The author reports no conflicts of interest. The author alone is responsible for the content and writing of the paper.

REFERENCES


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Research Paper

Value-at-risk bounds for multivariate heavy tailed distribution: an application to the Glosten–Jagannathan–Runkle generalized autoregressive conditional heteroscedasticity model

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ABSTRACT

The aim of this paper is to derive value-at-risk (VaR) bounds for the portfolios of possibly dependent financial assets for heavy tailed Glosten–Jagannathan–Runkle generalized autoregressive conditional heteroscedasticity processes using extreme value theory copulas. Using the 2014 contribution of Gammoudi et al made in “Value at risk estimation for heavy tailed distributions” as well as the 2005 paper by Mesfioui and Quessy titled “Bounds on the value-at-risk for the sum of possibly dependent risks”, we provide modified VaR bounds for when a shift of location is introduced.
These bounds have the interesting property of location invariance. Empirical studies for several market indexes are carried out to illustrate our approach.

Keywords: Glosten–Jagannathan–Runkle generalized autoregressive conditional heteroscedasticity (GJR GARCH); copulas; portfolio value-at-risk (pVaR); dependent risks; backtesting.

1 INTRODUCTION

Value-at-risk (VaR) has become a focal point for anyone thinking about financial risk. It has even become the cornerstone of many risk management frameworks. From a statistical viewpoint, this measure is nothing more than a given percentile of the profit-and-loss distribution over a fixed horizon. VaR techniques aim to quantify the worst expected loss of a portfolio over a specific time interval at a given confidence level $\alpha$. In other words, VaR is the highest $\alpha$-quantile of potential losses that can occur within a given portfolio during a specified time period. Although VaR is widely adopted by the financial community, it has undesirable mathematical characteristics, such as a lack of subadditivity and convexity. So, VaR is not a coherent measure of risk for non-elliptical portfolios (see Artzner et al 1999). Hence, the risk associated with a given portfolio may be greater than the sum of the risks of the individual assets measured by VaR. Generally, VaR is computed under the assumption of conditional normality; however, asset returns usually come from fat-tailed distributions. Also, overlooking stylized facts such as heteroskedasticity, volatility clustering and the leverage effect can be a significant source of error. It is often observed that for most financial time series volatilities tend to cluster together. To model this problem, Engle (1982) and Bollerslev (1986) have proposed the autoregressive conditional heteroscedasticity (ARCH) and generalized ARCH (GARCH) models. Based on the influence of asymmetric effects on the accuracy of VaR estimates, Brooks and Persand (2003) proved that models that do not allow for asymmetries in the volatility specification underestimate the true VaR. A variety of volatility models have been developed to accommodate this asymmetry in the response. These include the Glosten–Jagannathan–Runkle generalized autoregressive conditional heteroscedasticity (GJR GARCH), exponential GARCH (EGARCH) and quadratic GARCH (QGARCH) models. Earlier literature on inference from GARCH models is based on maximum likelihood estimation (MLE), assuming that the innovations follow a Gaussian distribution. However, plenty of empirical evidence has documented heavy tailed and asymmetric distributions of innovation series. In addition, dependence seems to be particularly pronounced during stock market crises, which emphasizes that financial assets become more dependent in the lower tail during extreme market movements (see Longin and Solnik 2001; Ang and Chen 2002; Cappiello et al 2006). The key to measuring an aggregate position risk is to accurately capture the underlying dependence structure between the
marginal business lines. In the context of elliptical distribution, the correlation matrix is naturally used to describe the dependence structure of the individual risks. However, outside the family of elliptical distributions, correlation only provides partial and often misleading information about the actual underlying dependencies (see Embrechts et al 1999). Copulas seem to be an interesting alternative for dependence characterization (see Nelsen (1999) for an excellent exposition). In order to estimate the portfolio VaR in a classical way, one has to assume a marginal return distribution and a specific dependence structure between the portfolio components by setting parametric copulas to data (see, for example, Mendes and Souza 2004; Miller and Liu 2006; Ozun and Cifter 2007). This may lead to a model-dependent view of stochastic dependence. A more flexible and reliable approach consists of estimating portfolio risk when there is no or partial information on the asset’s dependence structure, and only their marginal distributions are known. It is obvious in this specific framework (unknown dependence structure) that an explicit portfolio VaR is unreachable, but we are interested in finding lower and upper bounds on VaR. Recent contributions include Denuit et al (1999), Luciano and Marena (2002), Denuit et al (2005), Embrechts et al (2005), Mesfioui and Quessy (2005), Embrechts and Puccetti (2006a) and Kaas et al (2008). We refer the reader to Embrechts and Puccetti (2006b) for an extension to the general case of different marginal distributions, where a numerical procedure is suggested to compute VaR bounds. As mentioned in Slim et al (2012), numerous works have suggested a numerical procedure to compute VaR bounds. In this paper, we attempt to give explicit VaR bounds for the sum of possibly dependent financial assets modeled using nonlinear models (GJR GARCH). Our choice of the GJR GARCH model is justified due to its usefulness in detecting stylized facts; in particular, it exhibits the leverage effect. Thereafter, extreme value theory copulas (Copula–EVT) are used as a method for modeling and measuring extremes of the innovation series. Then, we try to modify the VaR bounds in order to take into consideration a large shift of location.

The remainder of this paper is organized as follows. Section 2 recalls the notion of portfolio VaR and some mathematical background about the concept of copulas and the Copula–VaR-bound approach. Section 3 offers a brief description of the GJR GARCH model and the proposed method. A real case study will be treated in Section 4, and, finally, a conclusion is made in Section 5.

2 VALUE-AT-RISK BOUNDS BASED ON THE COPULA CONCEPT

In this section, we discuss the basic concept of copulas and the fundamental results about the problem of bounding VaR for functions of dependent risks. We refer the interested reader to Nelsen (1999) for a more detailed treatment.
I. Gammoudi et al

**Definition 2.1** Let \( f : \mathbb{R} \to \mathbb{R} \) be a nondecreasing function. Its generalized left-continuous inverse is the mapping \( f^{-1} : \mathbb{R} \to \mathbb{R} \), defined by

\[
f^{-1}(t) = \inf\{s \in \mathbb{R} \mid f(s) \geq t\}.
\]

The VaR at probability level \( \alpha \in [0, 1] \) for a random variable \( X \) with a (right-continuous) distribution function \( F \) is defined by

\[
\text{VaR}_\alpha = F^{-1}(\alpha).
\]

In order to evaluate the risk level of a portfolio of possibly dependent financial assets (aggregate position), one must involve the dependence structure among the risks. This can be carried out through the use of copulas. The concept of the copula was introduced by Sklar (1959), but only recently has its potential for applications in finance become clear. A detailed treatment of copulas along with their relationship with different concepts of dependence is given by Joe (1997) and Nelsen (2006). A review of applications of copulas to finance can be found in Embrechts et al (2003) and Cherubini et al (2004).

Let the multivariate distribution function of a random vector \( X = (X_1, \ldots, X_d) \) be defined as \( H(x_1, \ldots, x_d) = \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) \), and denote by \( F_1, \ldots, F_d \) the set of associated marginal distributions. The theorem of Sklar (1959) states that there exists a multidimensional copula \( C \) such that \( H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \) for all \( (x_1, \ldots, x_d) \in \mathbb{R}^d \). For continuous marginal distributions, \( C \) is unique. Therefore, the multivariate distribution function \( H \) is linked to the marginal distributions via the copula function \( C : [0, 1]^d \to [0, 1] \). Let us denote the risky asset losses with \( X_1, \ldots, X_d \), where \( X_i : \Omega \to \mathbb{R}, i = 1, 2, \ldots, d \). Also consider the portfolio strategy consisting of investing a fixed relative amount \( w_i \in [0, 1] \) of the capital in the \( i \)th asset (short sales being excluded), so that \( \sum_{i=1}^{d} w_i = 1 \) (the portfolio is fully invested). For given weights, the portfolio loss can be expressed as the sum of the random variable \( S_i = w_i X_i \) with known continuous marginal distribution functions \( F_1, \ldots, F_d \). It is assumed throughout this chapter that the underlying dependence structure of \( S_1, \ldots, S_d \) described by the copula \( C \) is unknown. However, it is supposed that there exist copulas \( C_L \) and \( C_U \) such that \( C \geq C_L \) and \( C \geq C_U \).

Let us denote by \( F_S \) the distribution function of the portfolio loss, \( S = S_1 + \cdots + S_d \). For high-dimensional portfolios \( (d > 2) \), stochastic bounds for \( F_S \), easily obtained from the multivariate version presented in Cossette et al (2002) of the Makarov (1981) result, are \( F_L(s) \leq F_S(s) \leq F_U(s) \), where

\[
F_L(s) = \sup_{u_1 + \cdots + u_d = s} C_L\{F_1(u_1), \ldots, F_d(u_d)\}
\]

and

\[
F_U(s) = \inf_{u_1 + \cdots + u_d = s} C_U\{F_1(u_1), \ldots, F_d(u_d)\}.
\]
The distribution functions $F_L$ and $F_U$, known respectively as the lower and upper Fréchet bounds, provide the best possible bounds of $F_S$ in terms of stochastic dominance (see Lehmann 1966). For $d = 2$, the upper and lower bounds are themselves copulas. However, for a higher dimension ($d > 2$), the lower bound $F_L$ is not a distribution function any more (see Embrechts et al 2003). When no information about the dependence structure of $d$ risks is available, and if for each $i \in \{1, \ldots, d\}$ there exists a number $s_i^*$ such that the density $f_i(s)$ is non-increasing for all $s \leq s_i^*$, it is shown by Mesfioui and Quessy (2005, Proposition 3.3 and Theorem 3.1) as a special case of the Embrechts et al (2003) result that portfolio VaR for a given confidence level $\alpha$, denoted by $\text{VaR}_\alpha(S)$, is bounded as follows:

$$\text{VaR}(\alpha) \leq \text{VaR}_\alpha(S) \leq \overline{\text{VaR}}(\alpha), \quad (2.3)$$

where

$$\overline{\text{VaR}}(\alpha) = \inf_{u_1 + \cdots + u_d = \alpha + d-1} \sum_{i=1}^{d} F_i^{-1}(\alpha_i) \quad (2.4)$$

and

$$\text{VaR}(\alpha) = \max_{1 \leq i \leq d} \{ F_i^{-1}(\alpha) + \sum_{1 \leq j \neq i} F_j^{-1}(0) \}, \quad (2.5)$$

for $\alpha \leq \min\{F_1(s_1^*), \ldots, F_n(s_n^*)\}$. Gammoudi et al (2012) proposed another approach to computing explicit VaR bounds for largest order statistics, which are a limiting result of an infinity shift of location of the estimators’ VaR bounds defined in (2.4) and (2.5). For the largest order statistic, the modified upper and lower VaR bounds, denoted respectively by $\overline{\text{VaR}}^*(\alpha)$ and $\text{VaR}^*(\alpha)$, with $\frac{1}{2} < \alpha < 1$, are given by

$$\overline{\text{VaR}}^*(\alpha) = \sum_{i=1}^{d} \left\{ S_i(n-m_i) + D_i \left( \frac{m_i}{Tq_i(\alpha)} \right) \frac{1}{m_i} \sum_{j=1}^{m_i} (S_i(n-j+1) - S_i(n-m_i)) \right\}$$

and

$$\text{VaR}^*(\alpha) = \sum_{i=1}^{d} S_i(n-m_i) + \max_{1 \leq i \leq d} \left\{ D_i \left( \frac{m_i}{n(1-\alpha)} \right) \frac{1}{m_i} \sum_{l=1}^{m_i} (S_i(n-l+1) - S_i(n-m_i)) \right\} + \sum_{1 \leq j \neq i \leq d} \left\{ D_j \left( \frac{m_j}{n} \right) \frac{1}{m_j} \sum_{l=1}^{m_j} (S_j(n-l+1) - S_j(n-m_j)) \right\}.$$

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where
\[ q_i(\alpha) = \left\{ \frac{(1 - \alpha) \prod_{i=1}^{d}(G_{i+1}) L_i}{\sum_{i=1}^{d} |L_i|} \right\}^{1/(G_{i+1})} \]

with
\[ D_i \left( \frac{m_i}{n q_i(\alpha)} \right) = \frac{(m/n q_i(\alpha))^{G_{in}} - 1}{G_{in}} (1 - \min(0, G_{in})) , \]

and
\[ G_{in} = 1 - \frac{1}{2(1 - Q_{in})} \]

with
\[ Q_{in} = \frac{\left[ (1/m_i) \sum_{j=1}^{m_i} (S_{i(n-j+1)} - S_{i(n-m_i)}) \right]^2}{(1/m_i) \sum_{j=1}^{m_i} (S_{i(n-j+1)} - S_{i(n-m_i)})^2} . \]

The VaR bounds proposed by Gammoudi et al (2012) did not take into consideration the stylized facts of financial data, in particular the volatility clustering and the leverage effect. To deal with this problem, the following section proposes using the EVT/GJR GARCH combination to calculate modified VaR bounds where a shift of location is introduced.

3 VALUE-AT-RISK BOUNDS BASED ON THE GLOSTEN–JAGANNATHAN–RUNKLE GENERALIZED AUTOREGRESSIVE CONDITIONAL HETEROSCEDASTICITY MODEL AND EXTREME VALUE THEORY COPULAS

First, a specific GJR GARCH model is introduced in order to take the asymmetric effect into consideration. Second, VaR bounds for this kind of time series model are proposed. There is quite an extensive amount of literature documenting the behavior of stock returns’ volatility in stock markets. Thus, different estimating approaches around the volatility problem have been developed. Two widespread approaches are ARCH and GARCH, devised by Engle (1982) and Bollerslev (1986), respectively. The various variants of GARCH models that have emerged since these GARCH approaches have failed to capture the asymmetric features of returns behavior. Therefore, recent related studies have attempted to develop asymmetric GARCH approaches in order to capture the asymmetric volatility and promote the predictability of financial derivatives (Ding et al 1993; Duan 1995; Engle and Ng 1993; Fornari and Mele 1997; Hentschel 1995; Nelson 1991; Pagan and Schwert 1990; Sabbatini and Linton 1998; Szakmary et al 2003; Zakoian 1994). The GJR GARCH model was introduced by Glosten et al (1993). It is a nonlinear GARCH model that allows the asymmetries in response to the conditional variance on an innovation to be taken into account. The logic of this model is similar to that of regime-switching models, specifically
the threshold models. The principle of the GJR GARCH model is that the dynamics of conditional variance admits that a regime switch depends on the sign of past innovations.

The GJR GARCH (1,1) specification is given by

\[ r_{in} = \mu + \epsilon_{in}, \]  
\[ \epsilon_{in} = \sigma_{in} Z_{in}, \]  

with

\[ \sigma_{in}^2 = \omega_i + \beta_i \sigma_{i(n-1)}^2 + \alpha_i \epsilon_{i(n-1)}^2 + \lambda I_{i(n-1)} \epsilon_{i(n-1)}^2, \]  

where \( r_{in} \) is the asset return at \( n \) and \( \sigma_{in} \) is the volatility. Where the process is well defined if these conditions are fulfilled, \( \omega_i > 0, \alpha_i > 0, \beta_i > 0 \) and \( Z_{in} \) are independent and identically distributed (iid) innovations. \( I_{i(n-1)} \) is a dummy variable, where

\[ I_{\epsilon_i(n-1) \leq 0} = \begin{cases} 1 & \text{if } \epsilon_i(n-1) \leq 0, \\ 0 & \text{else}. \end{cases} \]  

This model can be written, after successive iterations, as

\[ \sigma_{in}^2 = \omega_i \left\{ 1 + \sum_{\tau=1}^{\infty} \prod_{j=1}^{\tau} \left( \alpha_i + \frac{\lambda_i}{2} I_{Z_{i(n-j)} \leq 0} \right) Z_{i(n-j)}^2 + \beta_i \right\}. \]  

Thereby, the \( Z_{i(n-1)}^2 \) has a different effect on the condition variance \( \sigma_{in}^2 \). So, \( I_{i(n-1)} Z_{i(n-1)}^2 \) is the squared value of negative shocks. Using the straightforward application of VaR bounds defined in (2.4) and (2.5) for the special case of a GJR GARCH generalized extreme value (GEV) distribution for marginal asset extreme losses, we can formulate the following proposition.

**Proposition 3.1** Define \( (S_i)_{1 \leq i \leq d} \) as a random variable with an absolutely continuous probability distribution function \( F_Y \). Suppose that \( m_i/(n(1-\alpha)) \geq 1 \). When no information is available about the dependence structure of \( (S_1, \ldots, S_d) \), then the non-modified upper and lower VaR bounds under the GJR GARCH model, denoted respectively by \( \text{VaR}^{\text{NM}}_{\text{GJR}}(\alpha) \) and \( \text{VaR}^{\text{NM}}_{\text{GJR}}(\alpha) \), with \( \frac{1}{2} < \alpha < 1 \), are given by

\[ \text{VaR}^{\text{NM}}_{\text{GJR}}(\alpha) = \sum_{i=1}^{d} \sigma_{in} \left[ Z_{i(n-m_i)} + \frac{(m_i/(n q_i(\alpha)))\hat{y}_{in} - 1}{\hat{y}_{in}} (1 - \min(0, \hat{y}_{in})) Z_{i(n-m_i)} M_{i(n)} \right] \]  

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\[ \text{VaR}^{\text{NM}}_{\text{GJR}}(\alpha) = \sum_{i=1}^{d} \sigma_{in} \left( Z_{i(n-m_i)} \right) + \max_{1 \leq i \leq d} \frac{(m_i/n(1-\alpha))\hat{\gamma}_{in} - \hat{\gamma}_{in}}{\hat{\gamma}_{in}} (1-\min(0, \hat{\gamma}_{in})) Z_{i(n-m_i)} M_{in}^{(1)} \] 
\[ + \sum_{1 \leq j \neq i \leq d} \sigma_{jn} \left( \frac{(m_j/n)\hat{\gamma}_{jn} - 1}{\hat{\gamma}_{jn}} (1-\min(0, \hat{\gamma}_{jn})) Z_{j(n-m_i)} M_{jn}^{(1)} \right), \tag{3.7} \]

where

\[ q_i(\alpha) = \frac{C_i^{1/\hat{\gamma}_{in}+1} (1-\alpha)}{\sum_{i=1}^{d} C_i^{1/\hat{\gamma}_{in}+1}} \]

and \( C_i = (m_i/n)\hat{\gamma}_{in} (\min(0, \hat{\gamma}_{in}) - 1) Z_{i(n-m_i)} M_{in}^{(1)} \).

As outlined in Section 2, these bounds will be referred to as standard bounds of VaR. The quantile estimator based on the work of Dekkers et al. (1989) depends heavily on the actual location because of the steepness of the log function. We also investigate in this section the limiting values for VaR bounds when the shift goes to infinity. Then, we get the modified upper and lower bound estimations under the GJR GARCH model for the quantile of the sum \( S = S_1 + \cdots + S_d \); these are invariant with respect to location changes, where \( S_{in} = w_{in} r_{in} \) and \( \sum_{i=1}^{d} w_{in} = 1 \) (ie, the portfolio is fully invested). The results are given in Proposition 3.2.

**Proposition 3.2** Define \( (S_{in}^*)_{1 \leq i \leq d} = (L_{in}^* (S_{in} + K))_{1 \leq i \leq d} \), where \( S_{in} \) is a random variable with an absolutely continuous probability distribution function \( F_{\gamma} \),

\[ L_{in}^* = \frac{1}{\sum_{j=1}^{m_i} (Z_{i(n-j+1)} - Z_{i(n-m_i)})} \sum_{j=1}^{m_i} \ln \frac{Z_{i(n-j+1)} + K}{Z_{i(n-m_i)} + K}, \]

and \( K \) is a constant. Suppose that \( m_i/(n(1-\alpha)) \geq 1 \). When no information is available about the dependence structure of \( (S_1, \ldots, S_d) \), and if \( K \rightarrow \infty \), then the modified upper and lower VaR bounds under the GJR GARCH model, denoted respectively by \( \text{VaR}^{M}_{\text{GJR}}(\alpha) \) and \( \text{VaR}^{M}_{\text{GJR}}(\alpha) \), with \( \frac{1}{2} < \alpha < 1 \), are given by

\[ \text{VaR}^{M}_{\text{GJR}}(\alpha) = \sum_{i=1}^{d} A_i \left\{ Z_{i(n-m_i)} + D_i \left( \frac{m_i}{n q_i(\alpha)} \right) \frac{1}{m_i} \sum_{j=1}^{m_i} (Z_{i(n-j+1)} - Z_{i(n-m_i)}) \right\}, \]
The proof of Proposition 3.2 as well as the standard VaR bounds defined in (3.6) and (3.7) are given in the online appendix.

4 REAL CASE STUDY

In this section, the proposed VaR bound estimations are applied to five stock market indexes: DAX, CAC 40, NIKKEI 225, FTSE 100 and IGPA. Here, the data set is
TABLE 1  Descriptive statistics of the daily returns.

<table>
<thead>
<tr>
<th>Country (index)</th>
<th>Skew</th>
<th>Kurtosis</th>
<th>Std</th>
<th>Min</th>
<th>Max</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>Germany (DAX 30)</td>
<td>0.33</td>
<td>8.3724</td>
<td>0.6302</td>
<td>-4.2869</td>
<td>4.6893</td>
<td>0.0333</td>
</tr>
<tr>
<td>France (CAC 40)</td>
<td>0.15</td>
<td>8.2062</td>
<td>0.6091</td>
<td>-4.1134</td>
<td>4.6012</td>
<td>0.0146</td>
</tr>
<tr>
<td>Japan (NIKKEI 225)</td>
<td>0.06</td>
<td>8.1783</td>
<td>0.6734</td>
<td>-5.2598</td>
<td>5.7477</td>
<td>-0.0065</td>
</tr>
<tr>
<td>UK (FTSE 100)</td>
<td>0.58</td>
<td>10.1583</td>
<td>0.4968</td>
<td>-4.0235</td>
<td>4.0755</td>
<td>0.0143</td>
</tr>
<tr>
<td>Chile (IGPA)</td>
<td>0.31</td>
<td>14.28</td>
<td>0.3336</td>
<td>-1.9116</td>
<td>2.2449</td>
<td>0.0046</td>
</tr>
</tbody>
</table>

Std denotes standard deviation.

updated from the previous section. It consist of daily data that encompass the period from March 15, 1991 to January 21, 2009. This period includes two remarkable crises: the Asian financial crisis of 1997 and the recent financial crashes of 2008. There are 4500 observations of daily prices, which are transformed into the daily return \( r_t \):

\[
r_t = 100 \ln \left( \frac{P_t}{P_{t-1}} \right) .
\]  

The five stock market indexes are depicted in Figure 1. This figure shows that there is some high volatility after the crisis in 1997 and the subprime crisis in 2008, followed by periods of relative tranquility. Throughout Figure 1, it can be readily seen that the volatility concentrates itself in clusters, which is reflected in the significant correlations of squared returns.

For more details about the return series characteristics, descriptive statistics are given in Table 1.

From Table 1, we can see that the kurtosis coefficients for all indexes exceed the value of 3 found for the normal distribution. All of the indexes produce positive skewed coefficients, where a positive skew indicates that the tail on the right side is longer than on the left side. From Table 1 and Figure 2, it seems that the underlying returns can be characterized by heteroscedasticity and time-varying autocorrelation. The dependence of the cited indexes taken in pairs is depicted in Figure 2. As illustrated in the last section, Figure 2 shows the presence of a dependence structure for all stock market returns. This structure is persistent for both positive and negative returns. In Figure 2, it is clear that all scatter plots show that the underlying indexes are more or less strongly dependent.

An AR(1)–GJR GARCH(1,1) model is filtered for each stock market index to remove autocorrelation and capture the conditional heteroscedasticity by incorporating asymmetric leverage effects for volatility clustering. The reason for using a nonlinear autoregressive model for the five indexes is explained by Figure 3(i) and Figure 3(ii).
Figure 3(i) depicts the sample autocorrelation function (ACF) of standardized residuals on each index, while Figure 3(ii) illustrates the sample ACF of squared standardized residuals. The residuals are the iid(0,1) process upon which EVT estimation of the sample cumulative distribution function (CDF) tails is based. The nonlinear model is supported by a Q-test, where squared returns are used. This test confirms that there is autocorrelation in the second moment. Additionally, we run Engle’s ARCH test, which confirms the presence of ARCH effects in our time series. We fitted the asymmetric model as

\[ r_{in} = c + \phi_i r_{i(n-1)} + Z_{in}, \]

\[ \varepsilon_{in} = \sigma_{in} Z_{in}, \quad \text{where} \quad \sigma_{in}^2 = \omega_i + \alpha_i \varepsilon_{i(n-1)}^2 + \psi_i I_{\varepsilon_{i(n-1)}<0} \varepsilon_{i(n-1)}^2 + \beta_i \sigma_{i(n-1)}^2, \]

(4.2)

(4.3)
FIGURE 2 Dependence structure of returns pairs.

(a) CAC 40/DAX. (b) CAC 40/NIKKEI 225. (c) CAC 40/FTSE 100. (d) CAC 40/IGPA. (e) DAX/NIKKEI 225. (f) DAX/FTSE 100. (g) DAX/IGPA. (h) NIKKEI 225/FTSE 100. (i) NIKKEI 225/IGPA. (j) FTSE 100/IGPA.

and

\[ \epsilon_{in} = r_{in} - \mu_{in}, \quad Z_{in} \text{ is iid}(0, 1). \]  \hspace{1cm} (4.4)

In the model given by (4.3), we took a data set starting with low orders of GARCH and then selected the ones that fitted the data best. We tried models of different orders for each time series, and from those with significant coefficients we made the selection based on a likelihood ratio (LR) test. Two competing models were chosen: AR(1)–GJR GARCH(1,1) (unrestricted model) and GARCH(1,1) (restricted model). The LR test and the final parameters for each time series are summarized in Table 2.

As outlined in the last section, the estimation of the Dekkers et al (1989) tail index \( \hat{\gamma}_n \) and the modified tail index \( \hat{G}_n \) requires us to choose the number \( m_i, 1 < i < d \), of upper order statistics for each portfolio component \( S_1, \ldots, S_d \). It is well known that tail index estimators are very sensitive to the choice of the sample fraction \( m \). This
FIGURE 3  (i) Plots of the sample ACF of standardized residuals.

(a) CAC 40. (b) DAX. (c) NIKKEI 225. (d) FTSE 100. (e) IGPA.

problem has been the subject of intense research in theoretical statistics, and it is still developing. The optimal sample fraction $m$ is selected through the minimization of the asymptotic mean square error, as in Danielsson et al (2001) or Drees and Kaufmann (1998). In a general fashion, by looking at these indicators one can see that the single stock market index’s loss distributions are skewed and heavy tailed.

As outlined in the previous section, and as an illustration of the proposed approach for measuring the risk in the $d = 2$ case, we consider three equally weighted portfolios. The first portfolio (PF1) is composed of the French and German stock market indexes, while the second portfolio (PF2) contains the Japanese and Chilean stock market indexes. The third portfolio (PF3) is a five-dimensional portfolio including all of the stock market indexes listed in Table 2. In Table 3, Table 4 and Table 5, the upper and lower VaR bounds for PF1, PF2 and PF3 are displayed under the assumption of a GEV distribution for extreme marginal losses of stock market indexes as well as under the AR(1)–GJR GARCH(1,1) model. As concluded in the last section, given the positive dependence between CAC 40 and DAX returns, one is interested in the upper bounds of VaR and compares these bounds to the exact $\text{VaR}_\alpha(S)$ under the
assumption of comonotonicity (the so-called perfect positive dependence). It is clear that VaR bound estimates under the AR(1)–GJR GARCH(1,1) model for marginal losses are close to the empirical quantiles for the different confidence levels between 0.95 and 0.99 for each portfolio.
TABLE 3 VaRs of PF1.

<table>
<thead>
<tr>
<th>α</th>
<th>VaR&lt;sub&gt;M_GJR&lt;/sub&gt;</th>
<th>VaR&lt;sub&gt;M_GJR&lt;/sub&gt;</th>
<th>Historical VaR</th>
<th>VaR&lt;sub&gt;NM_GJR&lt;/sub&gt;</th>
<th>VaR&lt;sub&gt;NM_GJR&lt;/sub&gt;</th>
<th>VaR</th>
<th>VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>0.018875</td>
<td>0.027032</td>
<td>0.026475</td>
<td>0.017661</td>
<td>0.029832</td>
<td>0.0114213</td>
<td>0.029452</td>
</tr>
<tr>
<td>0.97</td>
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<td>0.036014</td>
<td>0.034176</td>
<td>0.025817</td>
<td>0.039457</td>
<td>0.0191213</td>
<td>0.040401</td>
</tr>
<tr>
<td>0.99</td>
<td>0.039784</td>
<td>0.049564</td>
<td>0.048276</td>
<td>0.039512</td>
<td>0.049933</td>
<td>0.0292013</td>
<td>0.052132</td>
</tr>
</tbody>
</table>

Modified VaR bounds under the GJR GARCH model (VaR<sub>M_GJR</sub>, VaR<sub>M_GJR</sub>), non-modified VaR bounds under the GJR GARCH model (VaR<sub>NM_GJR</sub>, VaR<sub>NM_GJR</sub>) and non-dynamic VaR bounds (VaR, VaR).

TABLE 4 VaRs of PF2.

<table>
<thead>
<tr>
<th>α</th>
<th>VaR&lt;sub&gt;M_GJR&lt;/sub&gt;</th>
<th>VaR&lt;sub&gt;M_GJR&lt;/sub&gt;</th>
<th>Historical VaR</th>
<th>VaR&lt;sub&gt;NM_GJR&lt;/sub&gt;</th>
<th>VaR&lt;sub&gt;NM_GJR&lt;/sub&gt;</th>
<th>VaR</th>
<th>VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
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<td>0.021325</td>
<td>0.015385</td>
<td>0.007981</td>
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<td>0.031201</td>
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<td>0.019822</td>
<td>0.018921</td>
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<td>0.057921</td>
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</table>

Modified VaR bounds under the GJR GARCH model (VaR<sub>M_GJR</sub>, VaR<sub>M_GJR</sub>), non-modified VaR bounds under the GJR GARCH model (VaR<sub>NM_GJR</sub>, VaR<sub>NM_GJR</sub>) and non-dynamic VaR bounds (VaR, VaR).

TABLE 5 VaRs of PF3.

<table>
<thead>
<tr>
<th>α</th>
<th>VaR&lt;sub&gt;M_GJR&lt;/sub&gt;</th>
<th>VaR&lt;sub&gt;M_GJR&lt;/sub&gt;</th>
<th>Historical VaR</th>
<th>VaR&lt;sub&gt;NM_GJR&lt;/sub&gt;</th>
<th>VaR&lt;sub&gt;NM_GJR&lt;/sub&gt;</th>
<th>VaR</th>
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<tr>
<td>0.95</td>
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<td>0.016581</td>
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<td>0.019894</td>
<td>0.017615</td>
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<tr>
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<td>0.020130</td>
<td>0.0492317</td>
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</tbody>
</table>

Modified VaR bounds under the GJR GARCH model (VaR<sub>M_GJR</sub>, VaR<sub>M_GJR</sub>), non-modified VaR bounds under the GJR GARCH model (VaR<sub>NM_GJR</sub>, VaR<sub>NM_GJR</sub>) and non-dynamic VaR bounds (VaR, VaR).

4.1 Backtesting

This section evaluates the bounds provided in Section 3 through backtesting. We considered all of the equally weighted portfolios of five indexes, as indicated in Section 4. In this backtesting study, the values of m for each index are taken as given in Table 2. For the VaR bounds accuracy test, unconditional coverage tests were performed for our proposed models, NM (non-modified) and non-dynamic VaR. The results are summarized in Tables 6–8.

Following Tables 6–8, it is clear that the proposed approach based on the combination of GJR GARCH and Copula–EVT is more appropriate for calculating risk measures at the 1 and 5% levels for each portfolio. From Tables 6 and 7, it is clear that the non-dynamic VaR and non-modified VaR are no competition for the proposed
TABLE 6 Backtesting of the proposed VaRs for PF1.

<table>
<thead>
<tr>
<th>Kupiec test (p-value)</th>
<th>( \alpha )</th>
<th>( \text{VaR}^M_{\text{GJR}} )</th>
<th>( \text{VaR}^M_{\text{M}} )</th>
<th>( \text{VaR}^M_{\text{NM}} )</th>
<th>( \text{VaR}^M_{\text{NM}} )</th>
<th>( \text{VaR} )</th>
<th>( \text{VaR} )</th>
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<tr>
<td>0.95</td>
<td>0.231</td>
<td>0.331</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.99</td>
<td>0.209</td>
<td>0.221</td>
<td>0.197</td>
<td>0.210</td>
<td>0.089</td>
<td>0.093</td>
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</tbody>
</table>

TABLE 7 Backtesting of the proposed VaRs for PF2 (values of p-value).

<table>
<thead>
<tr>
<th>Kupiec test (p-value)</th>
<th>( \alpha )</th>
<th>( \text{VaR}^M_{\text{GJR}} )</th>
<th>( \text{VaR}^M_{\text{M}} )</th>
<th>( \text{VaR}^M_{\text{NM}} )</th>
<th>( \text{VaR}^M_{\text{NM}} )</th>
<th>( \text{VaR} )</th>
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<td>0.314</td>
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<td>0.99</td>
<td>0.219</td>
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<td>0.092</td>
<td>0.111</td>
<td>0.071</td>
<td>0.091</td>
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</tbody>
</table>

TABLE 8 Backtesting of the proposed VaRs for PF3 (values of p-value).

<table>
<thead>
<tr>
<th>Kupiec test (p-value)</th>
<th>( \alpha )</th>
<th>( \text{VaR}^M_{\text{GJR}} )</th>
<th>( \text{VaR}^M_{\text{M}} )</th>
<th>( \text{VaR}^M_{\text{NM}} )</th>
<th>( \text{VaR}^M_{\text{NM}} )</th>
<th>( \text{VaR} )</th>
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<tr>
<td>0.95</td>
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<td>0.135</td>
<td>0.081</td>
<td>0.0991</td>
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<td>0.99</td>
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<td>0.117</td>
<td>0.123</td>
<td>0.032</td>
<td>0.042</td>
<td></td>
</tr>
</tbody>
</table>

one; for example, at 5%, the null hypothesis of the non-occurrence model is accepted for the upper and lower bounds, but in Table 8 we should verify the conventional assumption that diversification can reduce the risk. That is why all three models are accepted here.

5 CONCLUSION

In this paper, explicit VaR bounds for portfolios of possibly dependent financial assets were evaluated. For all GARCH models, VaR bounds were proposed. This proposition incorporated a GJR GARCH model with the empirical copula to model the time-varying returns distribution. These standard bounds were obtained without imposing a specific assumption on the dependence structure between asset returns. The underlying dependence between the portfolio components was captured throughout the empirical copulas. In order to take into account the possibility of location changes, the standard bounds were evaluated when a large shift of location was applied to the
original data. The reverse translated bounds applied to the shifted data converged to the modified bounds, which enjoy the desirable properties of location invariance and scale invariance. The latter is a property that is also common to the classical Dekkers et al (1989) estimator in a multivariate setting. Empirical application and backtesting procedures were carried out to prove the competitiveness of our AR(1)–GJR GARCH VaR. However, the modified lower bound outperformed the standard lower bound when asset risks were independent or countermonotonic, but only for extremely high levels of probability.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

REFERENCES


Research Paper

Some options for evaluating significant deterioration under IFRS 9

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ABSTRACT

According to International Financial Reporting Standard 9 (IFRS 9), if the credit risk on an instrument has increased “significantly” since the instrument’s original recognition, and the resulting credit risk is more than “low credit risk”, then the institution would recognize a loss allowance on the instrument in the amount of lifetime expected credit losses (ECLs). Alternatively, at original recognition, and thereafter in the absence of significant deterioration in credit risk, the institution would recognize an allowance in the amount of twelve months of ECLs (or a lifetime, if the instrument matures in less than twelve months). In clarifying this aspect of IFRS 9, the International Accounting Standards Board (IASB) has specified that, in evaluating whether an instrument has suffered significant deterioration, an institution should consider only lifetime default risk, excluding consideration of possible changes in the exposure at default (EAD) and loss given default (LGD) components of ECLs. The authorities have also stated that the triggering of a lifetime allowance would reflect circumstances under which the spread inherent in contractual pricing no longer fully compensates for credit risk.

Further, in its analysis of IFRS 9, the Bank for International Settlements has presented a view that any deterioration in credit risk should be considered significant.
But how, precisely, does an institution determine whether an instrument has suffered a significant deterioration in its credit risk to a point in excess of low credit risk? Beyond the unspecific guidance mentioned above, the authorities have said little about this. As a result, many institutions appear to be unclear on this matter and remain provisional in their planning for full implementation of IFRS 9. In this paper, we present some implementation options that, for corporate and commercial portfolios of instruments evaluated individually, seem compatible with the IASB’s limited guidance to date. As a leading candidate for a decision measure, we introduce the levelized forward probability of default (LFPD). The LFPD translates a PD term structure into a spread equivalent. If this spread measure increases, one can conclude that the (real not risk-neutral) PD component of the instrument’s par spread has increased, and so, on the basis of default risk information alone, one would have evidence that the initial contract pricing was insufficient to cover prospective credit risk.

We present this default risk measure along with a couple of others; these similarly reduce an instrument’s PD term structure at original recognition and at a later reporting date to summary numbers, which one can rank and use in determining whether significant deterioration has occurred. Not all institutions will be able to calculate such PD-spread measures, at least initially, so we also review some alternative decision rules. We compare various options and suggest the use of a hierarchical structure based on an institution’s data availability, with the use of PD-spread measures at the top of the hierarchy. We describe ways to simplify the assessment of point-in-time (PIT) PD term structures. We explore ways of designing a threshold for stage 2 allocation and suggest combining a percentage-change threshold for significant deterioration together with an absolute threshold (on annualized default risk) for low credit risk. Last, we provide some perspective on implementation requirements for this standard.

**Keywords:** lifetime point-in-time (PIT) probability of default (PD); significant deterioration; stage allocation; International Financial Reporting Standard 9 (IFRS 9); expected credit loss (ECL).

## 1 OVERVIEW

The transition from current accounting standards, ie, International Accounting Standard 39 (IAS 39), to International Financial Reporting Standard 9 (IFRS 9) is underway, and one of the aspects with the least amount of guidance is the determination of significant deterioration. This requirement is part of the overall move from an incurred loss approach to expected loss and fair value-based approaches (Beerbaum 2015). IFRS (IFRS 2014a, 2014b) call for an institution to classify each of its financial instruments carried at amortized cost into one of three stages.
• Stage 1: performing. At initial recognition of a financial instrument, an institution should set aside an allowance in the amount of twelve months of expected credit losses (ECLs). This serves as a proxy for the ECLs initially priced into the instrument.

• Stage 2: significantly deteriorated. When credit quality deteriorates significantly to a position in excess of low credit risk, the allowance should change from twelve-month to lifetime ECLs.

• Stage 3: impaired.

IFRS (2014a, Section 5.5.10) defines the need for lifetime ECL for instruments that have deteriorated significantly:

At each reporting date, an entity shall measure the loss allowance for a financial instrument at an amount equal to the lifetime expected credit losses if the credit risk on that financial instrument has increased significantly since initial recognition.

Thus, determining whether, on an unimpaired instrument, the appropriate allowance amounts to a lifetime or twelve months of ECLs depends on whether the instrument has suffered significant deterioration.

IFRS (2014a, Section 5.5.9) defines the need for significant deterioration as follows:

At each reporting date, an entity shall assess whether the credit risk on a financial instrument has increased significantly since initial recognition. When making the assessment, an entity shall use the change in the risk of a default occurring over the expected life of the financial instrument instead of the change in the amount of expected credit losses. To make that assessment, an entity shall compare the risk of a default occurring on the financial instrument as at the reporting date with the risk of a default occurring on the financial instrument as at the date of initial recognition and consider reasonable and supportable information, that is available without undue cost or effort, that is indicative of significant increases in credit risk since initial recognition.

The above-mentioned requirement further leads to the following modeling and implementation requirements.

• “At each reporting date...” implies that the institution must be able to continuously reassess whether significant deterioration in lifetime default risk has occurred.

• “...credit risk on a financial instrument...” implies that the assessment has to be for each financial instrument.

• “...has increased significantly since initial recognition” implies that the comparison always involves an assessment of credit risk today in relation to the reference point of initial recognition.
...the change in risk of default...” implies that the assessment considers only the risk of default and not the loss given default (LGD) and exposure at default (EAD) components of ECL.

...occurring over the expected life...” implies the need to measure risk of default over the life of the instrument.

...instead of the change in the amount of expected credit...” reiterates that the determination of significant deterioration involves only measures of default.

...consider reasonable and supportable information...” places an onus on the institution to consider a wider range of relevant data in evaluating whether an instrument has experienced significant deterioration.

In our view, with these key requirements in mind, institutions now have the tasks of

- defining risk of default,
- assessing the definition of risk of default against the requirement,
- coming up with a business logic definition of risk of default that is customized to an institution’s portfolios and data availability,
- defining a mechanism of quantifying a significant deterioration,
- defining threshold levels to be used for the quantification of significant change, ie, how instruments get classified as stage 1 or stage 2,
- devising a robust implementation strategy to run this process in production.

This paper focuses on significant deterioration and is built around these key requirements and tasks. Since the topic is much debated, as there are almost no precedents guiding implementation of such a standard, we present some options for satisfying these requirements.

IFRS 9 requirements are relatively new, and while there are other papers, such as Reitgruber (2015) and Xu (2016), that focus on the calculation of ECL components, this paper focuses exclusively on significant deterioration. It does not delve into other IFRS 9 requirements, except for certain other standards that have a bearing on IFRS (2014a, Section 5.5.9), which introduce significant deterioration.

2 DEFINING RISK OF DEFAULT

We view the term “risk of default” as referring to the probability of default (PD) over one or more time intervals, with the coming year being the most common reference period. While we have seen others define the term in a more nebulous way, we always
focus on quantitative PDs. One can hardly imagine pricing for credit risk properly or managing a credit portfolio well unless one has numerical PDs. For further details on ways to quantify/represent risk of default, see IFRS (2012, 2013).

Credit institutions typically have one (and sometimes more than one) master scale expressing the relationship between ratings and PDs. Thus, henceforth, we refer to grades as (rounded) PDs. In the rare case of an institution without a master mapping table, the process of determining PDs from grades or of comparing grades from different grading models becomes more difficult.

The presence of PDs does not guarantee that an institution estimates default risk accurately. We find that many existing PD or grading models produce estimates that are biased, over- or understate realized default rates (DRs) on average, or are excessively stable, failing to track closely the temporal DR fluctuations, which can be large. The use of such PD estimates, taken at face value, would impair an institution’s management of credit risk. Thus, it makes sense to adjust PD models so that they are unbiased and point in time (PIT), tracking DRs as closely as possible. As a matter of terminology, we refer to a model as “hybrid” if its output is not fully PIT and not fully through the cycle (TTC). A TTC PD is a pro forma estimate under the usually counterfactual assumption that credit-cycle conditions are in a long-run average state.

Most analysts agree that IFRS 9 calls for the use of unbiased estimates of PDs over the life of a loan, with “lifetime PDs” interpreted as PD term structures. We elaborate on this in the following subsections.

### 2.1 One-year PIT PD

Even though IFRS (2014a, Section 5.5.9) clearly asks for change in lifetime risk of default, Sections 5.5.13 and 5.5.14 of IFRS 9 acknowledge that, under certain circumstances, change in one-year risk of default may be a reasonable proxy for change in lifetime risk of default:

The methods used to determine whether credit risk has increased significantly on a financial instrument since initial recognition should consider the characteristics of the financial instrument (or group of financial instruments) and the default patterns in the past for comparable financial instruments. Despite the requirement in paragraph 5.5.9, for financial instruments for which default patterns are not concentrated at a specific point during the expected life of the financial instrument, changes in the risk of a default occurring over the next twelve months may be a reasonable approximation of the changes in the lifetime risk of a default occurring. In such cases, an entity may use changes in the risk of a default occurring over the next twelve months to determine whether credit risk has increased significantly since initial recognition, unless circumstances indicate that a lifetime assessment is necessary.

However, for some financial instruments, or in some circumstances, it may not be appropriate to use changes in the risk of a default occurring over the next twelve
months to determine whether lifetime expected credit losses should be recognized.
For example, the change in the risk of a default occurring in the next twelve months may not be a suitable basis for determining whether credit risk has increased on a financial instrument with a maturity of more than twelve months when

(a) the financial instrument only has significant payment obligations beyond the next twelve months,
(b) changes in relevant macroeconomic or other credit-related factors occur that are not adequately reflected in the risk of a default occurring in the next twelve months, or
(c) changes in credit-related factors only have an impact on the credit risk of the financial instrument (or have a more pronounced effect) beyond twelve months.

The above IFRS passage seems paradoxical, since it allows the use of a one-year PD as a proxy for the lifetime PD only under circumstances in which the two are consistent in their depiction of default risk. One, however, can only know this by comparing the two alternatives, in which case one has the lifetime PD and therefore would not use the one-year PD as a proxy for lifetime. Nonetheless, we consider the use of one-year PDs as a possible option. Evidently, there are circumstances in which an institution has lifetime loss rates, to be used in determining lifetime ECLs, but, as yet, only one-year PDs. We accept that possibility, though it seems unlikely, and on that basis we will continue to discuss the one-year PD option.

In IFRS (2015), the Transition Resource Group clarifies the use of twelve-month versus lifetime PDs; we refer to this document on appropriateness, but, in this paper, we clarify the quantitative foundation of it.

In any case, regardless of whether one uses twelve-month or lifetime PDs in assessing significant deterioration or managing credit risk, one needs to make sure that the estimates are accurate. In our experience, the PD estimates for corporate and commercial portfolios are much too stable, failing to explain much of the temporal fluctuations in DRs. Since the significant deterioration standard involves a comparison of lifetime PDs at different points in time, the cross-temporal inconsistencies intrinsic to hybrid or TTC models make them ineligible for use in those comparisons. Hence, for IFRS 9, one needs to covert hybrid or TTC PD models to PIT ones.

The techniques for doing this draw either on realized DRs, observed in large samples and representative of a particular credit portfolio (Carlehed and Petrov 2012), or, as a way to mitigate sampling errors in DRs from anything other than very large portfolios, on summary measures of PDs obtained in turn from a PIT PD model (Aguais et al 2004, 2007; Forest et al 2013).

Here, we describe the second approach, which involves the use of industry–region credit-cycle Z indexes. One can find a detailed description of this approach in Forest et al (2015) and Chawla et al (2015, 2016).
Some options for evaluating significant deterioration under IFRS 9

Forest et al (2015) and Chawla et al (2015) find that agency ratings are roughly 80% TTC. They also demonstrate a way of converting those ratings to PIT PDs that closely track the temporal fluctuations in DRs of agency-rated companies. Equation (2.1) summarizes the way one might use industry–region credit-cycle indexes in converting hybrid or TTC PDs to PIT PDs.

\[
\text{PIT PD}_{i,t} = \Phi\left(-\frac{f(DD_{i,t}) + b\sqrt{\rho_{(i),R(i)}}Z_{1(i),R(i),t} + \sqrt{\rho_{(i),R(i)}}\Delta Z_{1(i),R(i),t}}{\sqrt{1-\rho_{(i),R(i)}}}\right),
\]

where PIT PD$_{i,t}$ is the PIT PD for the $i$th entity at time $t$; DD$_{i,t}$ is the idiosyncratic internal model score/DD/grade/rating for the $i$th entity at time $t$; $f$ is a bespoke functional form for every model, eg, logistic or probit; and $\Phi$ is the standard normal cumulative distribution function. $Z_{1(i),R(i),t}$, where “I” is industry and “R” is region, denotes the credit-cycle index (CCI) at time $t$. Z is a quantification of credit condition using the PIT–TTC framework. It measures how far an industry’s or region’s credit conditions are from its long-run average. $\Delta Z_{1(i),R(i),t+1}$ is the change in industry (I) and region (R) credit-cycle index (from $t$ to $t + 1$), $b$ is the regression coefficient that denotes the degree of TTC-ness of DD$_{i,t}$ and $\rho$ is the correlation factor related to DD_GAP.

Once one converts all PD model outputs to PIT, one can use these outputs in the intertemporal comparisons involved in evaluating significant deterioration.

By using industry–region credit-cycle indexes rather than broad macroeconomic indicators, one takes into account the occasionally large deviations in conditions across sectors. Figure 1 shows the conversion of a TTC BBB-rated entity’s grade to a PIT PD using the global oil and gas credit-cycle index. One can see from Figure 1 that the facilities to such entities originated in 2005–6 would have mostly suffered significant deterioration by 2008–9, as would be the case in most industries. However, we can also see that those originated to oil and gas firms in 2012–13 would have suffered significant deterioration by 2014–15, probably unlike most facilities in many other sectors. Thus, a broad macroeconomic indicator that gives the same signal for all sectors would fail to pick up such distinctions.

**2.2 Lifetime PIT PD term structure**

In writing this paper, we chose to start with a discussion of a short-term (eg, one-year) PD, because, to have any hope of building an accurate PD term structure, one must start with a good estimate of initial conditions. As discussed before, for corporate and commercial portfolios, the legacy models produce short-term PDs that are generally not PIT, and indeed far from it. Hence, one needs first to convert those outputs to PIT and then use those PIT estimates of initial conditions in producing PD term structures that extend over the life of each exposure.
There are techniques for doing this. One may start with a series of forward PD models. However, while this approach can produce PD term structures, it does not offer a way of running the joint, PD, LGD and EAD scenarios needed in accounting for the effects of PD, LGD and EAD correlation on the ECL term structures. So, the main alternative, which involves the use of transition matrixes in developing period-by-period PDs, is preferable. However, to do this in a way that accounts for current and prospective conditions, and not just long-run average conditions, one needs to use credit-cycle-conditional transition matrixes. One could imagine accomplishing this using macro-economic scenarios, as in existing regulatory stress-test models. But, at the present time, most institutions have not developed the capacity of generating the large number of probabilistic macroeconomic scenarios that one needs if one is to obtain not just current stresses but credible estimates of unconditional expectations extending several years. So, we instead use a more tractable and arguably more accurate approach, drawing on time-series models of the stochastic evolution of credit-cycle, Z indexes. Z-index scenarios entered into a CreditMetrics model of transition matrixes produce the needed transition and multi-period PD scenarios. The mathematics of multi-year term-structure construction is beyond the scope of this paper and deserves a full technical article in itself. Hence, we refrain from going into the details here, but simply summarize that the process leads to several future views of PIT PDs using Monte Carlo simulation of the future Z scenarios. The averages of the many PD scenarios provide estimates of the unconditional PIT PD term structure.
Figure 2 compares an unconditional lifetime PIT PD term structure of two entities, each with a TTC PD of 75 basis points (bps). Since global oil and gas is experiencing harsh credit conditions today, with news of bankruptcies and insolvencies, the one-year (four-quarter) PIT PD for the first entity in the oil and gas industry is higher than its TTC PD, and its cumulative PD term structure continues to climb higher. In stark contrast is the second entity, which has a TTC PD of 75bps but a lower one-year (four-quarter) PIT PD, with the overall term structure being lower compared with that of the first entity.

Since the standard calls for “risk of a default occurring over the expected life of the financial instrument”, it makes sense to look at the entire term structure of PD over the life of the exposure in evaluating significant deterioration.

Also, IFRS (2014a, Section 5.5.10) calls for lifetime ECL for facilities that have deteriorated significantly in credit risk:

> At each reporting date, an entity shall measure the loss allowance for a financial instrument at an amount equal to the lifetime expected credit losses if the credit risk on that financial instrument has increased significantly since initial recognition.

This means that institutions using PD, LGD and EAD modeling for ECL have to compute lifetime PIT PDs as part of the process in determining lifetime ECL. Hence, it makes sense to calculate the PDs together with ECLs.

The use of PIT PD term structures in gauging significant deterioration gives rise to a quantification question. A term-structure curve exists at the time of origination, and another exists at a later reporting date. How should the two be compared with the aim of determining which is higher and which is lower? We illustrate this problem in...
Figure 3, which considers the case of a facility with an original maturity of five years. Over two years, the cumulative PD term structure over the three years of remaining tenor has shifted up from the position of that same part of the term structure at original recognition, but the tenor has dropped by two years, causing some higher PD tenor points to fall out of the analysis. Has this facility suffered significant deterioration?

In assessing the curves (at time of origination and another one at reporting date) with the aim of determining whether significant deterioration has occurred, one must decide which of the two curves is “above” the other. To do this, one must apply a metric for reducing curves to numbers representing distances above the default-free, PD = 0 curve. We suggest the following three metrics.

1. Lifetime annualized PD (LAPD) expresses the lifetime PD as an annual average rate. This is a constant default-intensity rate that cumulates at term to the exposure’s lifetime PD. It involves no discounting of future incremental PDs, so it is always larger than the related LMPD or LFPD described below.

2. Levelized marginal PD (LMPD) is a weighted average of annualized marginal PDs, with discount rates driving from the effective interest rates (EIR) as weights. This is the constant, risk-free premium (rate) that, paid periodically up to the term of the exposure, cumulates to a present value (PV) that is the same as that of the exposure’s default risk while using the EIR in discounting.

3. Levelized forward PD (LFPD) is a weighted average of annualized forward PDs, with survival rates (SRs) multiplied by discount rates as weights. This...
is the constant premium (rate) that, paid periodically so long as the exposure survives, cumulates to a PV that is the same as that of the exposure’s default risk. Since the premium occurs only if the exposure survives, the LFPD must be larger than the corresponding LMPD.

LAPD, LMPD and LFPD provide similar measures of distances from risk-free equivalents. To derive these, we make use of a cumulative PD curve, marginal PD curve or forward PD curve, respectively, as shown in Figure 4. The vast majority of commercial and corporate exposures have (marginal) PD term structures that slope upward, revealing that annual average default risk rises with tenor. In these cases, therefore, if the risk grade remains fixed, the LAPD, LMPD and LFPD will decline as tenor shrinks. We illustrate these ways of measuring default risk over several time periods in Figure 4. Note that these metrics basically convert a PD term structure into a spread equivalent. Indeed, assuming that the real PD term structure is the same as the risk-neutral one, the LFPD corresponds to the par spread on a CDS contract that pays EAD in the event of default.

Since the LFPD represents a type of spread, this seems to align it with the IASB’s expressed intent that the significant deterioration trigger occurs under circumstances in which initial pricing no longer covers credit risk to the same extent as at original recognition. See, for example, the IFRS 9 project summary (IFRS 2014b, p. 20):

When credit is first extended the initial creditworthiness of the borrower and initial expectations of credit losses are taken into account in determining acceptable pricing and other terms and conditions. As such, recognizing lifetime expected credit losses from initial recognition disregards the link between pricing and the initial expectations of credit losses.

A true economic loss arises when expected credit losses exceed initial expectations (ie, when the lender is not receiving compensation for the level of credit risk to which it is now exposed). Recognizing lifetime expected credit losses after a significant increase in credit risk better reflects that economic loss in the financial statements.

The LAPD, LMPD and LFPD measures will indicate deterioration since origination only if a sufficiently large worsening of the risk grade or increase in the reference (eg, one-year) PD has occurred. “Sufficiently large” means “large enough to more than offset the risk reduction associated with shorter tenor”.

To implement the stage 1 versus stage 2 allocations, one needs to establish the amount of increase in the metric that would be considered significant. One could, for example, establish that a 10% increase in LFPD was significant, and that an LFPD over 20bps was in excess of “low credit risk”.

2.3 Other indicators representing risk of default

In some cases, particularly early on in the application of IFRS 9, when an institution may be lacking term structures at original recognition for many facilities, the
FIGURE 4 Visualizing (a) LAPD, (b) LMPD and (c) LFPD.

Institution may use other default risk indicators in evaluating whether significant deterioration has occurred. IFRS (2014a, Section B 5.5.17) provides a non-exhaustive list of indicators related to risk of default. We deliberately call them “indicators representing default” because these are not “pure quantifications” of risk of default. Most
Some options for evaluating significant deterioration under IFRS 9 are qualitative in nature and can be interpreted by people differently based on each individual’s experiences and biases. In Table 1, we show how these triggers can be applied to a large to medium-sized corporate portfolio. Such an example highlights the shortcomings of using such data to represent risk of default.

Further, most are backward looking, which defeats the very purpose of the forward-looking nature of IFRS 9. There are a variety of pros and cons when it comes to the use of such data. We believe that such data can be

- a good starting point when institutions do not have one-year PIT PD or lifetime PIT PD, but they cannot be a substitute because most of them are backward looking,
- a benchmark for one-year PIT PD or lifetime PIT PD (but only those indicators that are leading),
- a backstop in case forward-looking information is not available (for those indicators which are lagging, e.g., days past due).

3 SIGNIFICANT DETERIORATION THRESHOLDS

Significant deterioration requirements call for both a measure of lifetime default risk and thresholds to be used in determining whether the measure has increased significantly to a position in excess of low credit risk. There are some clear ways to establish thresholds.

- Absolute change: here, one may determine that the stage 2 threshold is exceeded if a facility’s lifetime PD measure increases by more than a fixed amount. For example, if one measures lifetime default risk using LFPD, one could set a threshold of 20bps. Thus, if the LFPD were to rise by more than 20bps, one would conclude that the facility had deteriorated significantly.

- Percentage change: in this case, one expresses the threshold as a percentage change. Again, using LFPD as the lifetime default measure, one would determine that a facility had deteriorated significantly if its LFPD had increased by more than a fixed percentage amount since the date of initial recognition. Suppose one sets a percentage threshold of 10%. Consider a facility with an initial LFPD of 70bps. In this case, one would conclude that the facility had suffered significant deterioration if its LFPD at a subsequent reporting date were to exceed 77bps. Alternatively, suppose that the initial LFPD was 50bps. In this case, a subsequent LFPD in excess of 55bps would indicate significant deterioration.
<table>
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<th>Assessing changes in credit risk by</th>
<th>Portfolio coverage</th>
<th>Coverage in terms of lifetime risk of default</th>
<th>Propensity to create false positives</th>
<th>Propensity to create false negatives</th>
<th>Timeliness</th>
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<tr>
<td>Internal price indicators of credit risk</td>
<td>High</td>
<td>Only if lifetime PD/grades or lifetime cashflows are used</td>
<td>Low</td>
<td>High due to TTC or hybrid grades, which do not reflect current credit conditions and forward forecast</td>
<td>Lagging if based on TTC or hybrid grades</td>
</tr>
<tr>
<td>Changes in rates or terms of existing instrument (such as forbearance)</td>
<td>High</td>
<td>High</td>
<td>Low</td>
<td>High: lagging nature means those deteriorated but not days past due are missed</td>
<td>Severely lagging</td>
</tr>
<tr>
<td>Significant changes in external market indicators</td>
<td>High</td>
<td>High</td>
<td>Low only if market signals are relevant drivers, otherwise creates noise</td>
<td>Low</td>
<td>Leading</td>
</tr>
<tr>
<td>External credit rating</td>
<td>Depends on portfolio coverage</td>
<td>High since ratings take a longer view</td>
<td>Low</td>
<td>High because external credit ratings are about 20% PIT and do not reflect current credit conditions and forward forecast</td>
<td>Somewhat lagging</td>
</tr>
<tr>
<td>Assessing changes in credit risk by</td>
<td>Portfolio coverage</td>
<td>Coverage in terms of lifetime risk of default</td>
<td>Propensity to create false positives</td>
<td>Propensity to create false negatives</td>
<td>Timeliness</td>
</tr>
<tr>
<td>--------------------------------------------------------------------------------------------------</td>
<td>--------------------</td>
<td>--------------------------------------------</td>
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<td>-------------------------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>An actual or expected internal credit rating</td>
<td>High</td>
<td>High if lifetime ratings are used</td>
<td>Low</td>
<td>High if TTC or hybrid grades, which do not reflect current credit conditions and forward forecast</td>
<td>Somewhat lagging</td>
</tr>
<tr>
<td>Existing or forecast adverse changes in business, financial or economic conditions</td>
<td>High</td>
<td>High</td>
<td>Low only if market signals are relevant drivers, otherwise creates noise</td>
<td>Low</td>
<td>Leading</td>
</tr>
<tr>
<td>An actual or expected significant change in the operating results of the borrower</td>
<td>Low: forecasting changes in operating results is hard and expensive</td>
<td>High</td>
<td>Low</td>
<td>Low</td>
<td>Leading</td>
</tr>
<tr>
<td>Significant increases in credit risk on other financial instruments of the same borrower</td>
<td>Low</td>
<td>High</td>
<td>Low</td>
<td>High: not all cases can be covered due to low coverage</td>
<td>Somewhat lagging</td>
</tr>
<tr>
<td>Assessing changes in credit risk by</td>
<td>Portfolio coverage</td>
<td>Coverage in terms of lifetime risk of default</td>
<td>Propensity to create false positives</td>
<td>Propensity to create false negatives</td>
<td>Timeliness</td>
</tr>
<tr>
<td>-----------------------------------</td>
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<td>--------------------------------------</td>
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</tr>
<tr>
<td>Actual or expected significant adverse change in the regulatory, economic or technological environment of the borrower</td>
<td>High</td>
<td>High</td>
<td>Low only if market signals are relevant drivers, otherwise creates noise</td>
<td>Low</td>
<td>Leading</td>
</tr>
<tr>
<td>Significant changes in the value of collateral supporting the obligation</td>
<td>Low: needs collateral valuation</td>
<td>Low</td>
<td>High: needs to create collateral valuation (LGD/EAD) to PD link</td>
<td>High: not all cases can be covered due to low coverage</td>
<td>Somewhat lagging</td>
</tr>
<tr>
<td>A significant change in the quality of the guarantee provided by a shareholder</td>
<td>Low: applies to guarantees only</td>
<td>Low</td>
<td>Low</td>
<td>High: not all cases can be covered due to low coverage</td>
<td>Somewhat lagging</td>
</tr>
<tr>
<td>Changes in the quality of credit enhancements</td>
<td>High</td>
<td>High</td>
<td>Low</td>
<td>High: lagging nature means those deteriorated but not days past due are missed</td>
<td>Somewhat lagging</td>
</tr>
<tr>
<td>Assessing changes in credit risk by</td>
<td>Portfolio coverage</td>
<td>Coverage in terms of lifetime risk of default</td>
<td>Propensity to create false positives</td>
<td>Propensity to create false negatives</td>
<td>Timeliness</td>
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</tr>
<tr>
<td>Expected changes in the loan documentation</td>
<td>High</td>
<td>High</td>
<td>Low</td>
<td>High: lagging nature means those deteriorated but not days past due are missed</td>
<td>Severely lagging</td>
</tr>
<tr>
<td>Changes in the payment status of borrowers in the group</td>
<td>Low</td>
<td>High</td>
<td>Low</td>
<td>High: lagging nature means those deteriorated but not days past due are missed</td>
<td>Somewhat lagging</td>
</tr>
<tr>
<td>Changes in the entity's credit management approach</td>
<td>Low</td>
<td>High</td>
<td>Low</td>
<td>High: not all cases can be covered due to low coverage</td>
<td>Somewhat leading</td>
</tr>
<tr>
<td>Past due information</td>
<td>High</td>
<td>High</td>
<td>Low</td>
<td>High: lagging nature means those deteriorated but not days past due are missed</td>
<td>Severely lagging</td>
</tr>
</tbody>
</table>

The propensity to create a false positive is like a type 1 error of a model. In this case, it refers to the data calling something significantly deteriorated when it is not significantly deteriorated, i.e., creating noise. The propensity to create false negatives is like a type 2 error of a model. In this case, it refers to the data calling something not significantly deteriorated when it is in fact significantly deteriorated, i.e., missing a relevant signal.
Absolute change with threshold rising in higher risk bands: this case is similar to the percentage change threshold.

Absolute level: this type of threshold applies in the cases of determining whether a facility has low credit risk. Again, consider using the LFPD for evaluating lifetime default risk. Here, one could use a value of, say, 30bps. In this case, if the LFPD at the reporting date were above 30bps, one would conclude that the facility’s default risk was greater than low default risk.

A combination of the above.

4 THRESHOLD LEVELS FOR CHANGE IN RISK OF DEFAULT

In the above mentioned trigger design, the PIT PD multiple largely covers most parts of the credit book, but the level of threshold is still an open question. Whether a 10%, 20% or some other percentage rise in the significant-deterioration metric is the proper threshold for significant deterioration is open to debate. There are no established conventions, but based on the advice of the Bank for International Settlements, the threshold will surely not be an extremely permissive one. Here, we focus on mathematical foundations for developing threshold levels rather than providing an outright solution or equation. Alternative ways of quantifying threshold levels include the following:

1. calibrating to banks’ existing credit processes, watch list criteria and acceptance criteria in order to identify the level of significant deterioration; however, this is marred with issues, such as lagging indicator and low coverage;

2. modeling errors whereby models’ error bounds are considered; however, the calculation of error bands for multi-year PDs is complex and restricted by limited data, so we doubt that this approach will be definitive;

3. basing the threshold on expert judgement, such as a level identified by a management review committee based on a general understanding of rating level changes, eg, a three PIT grade downgrade on a twenty-one-point scale is excessive; note that, in such a case, TTC-like grades will not work, eg, agency ratings of oil and gas and mining firms are not downgraded three or more notches, even though the sector is going through systematic distress.

5 IMPLEMENTATION CONSIDERATIONS

We think implementing significant deterioration in systems will be a challenge due to the following modeling-related reasons.
Stage allocation is done at the financial instrument level (not obligor level), leading to a cross-reference of systems and the possibility of instruments from the same obligor at different stages.

Historical data going back to origination may not be available. In the case of corporate and commercial portfolios, one can assume the same TTC-like grade and back-cast PIT PDs back in history, using the credit-cycle index approach discussed earlier, and thereby alleviating this concern somewhat.

Implementing a term structure for each entity and instrument, both historically and going forward in time, is time consuming and error prone.

A change of upstream models, eg, corporate model v1, implemented in 2012, v2 in 2015 and v3 in 2017 can make cross-model PD output comparison extremely hard.

Experts reviewing significant deterioration model output and attempting to override the model output by changing trigger or threshold levels can lead to an unstable and ever-changing model.

Reversion, ie, going back from stage 2 to stage 1, will have to be modeled and implemented as well.

IFRS (2014a, Section 5.5.7) provides the following guidance on reversion:

If an entity has measured the loss allowance for a financial instrument at an amount equal to lifetime expected credit losses in the previous reporting period, but determines at the current reporting date that paragraph 5.5.3 is no longer met, the entity shall measure the loss allowance at an amount equal to twelve-month expected credit losses at the current reporting date.

Since PDs are continuous but the stage allocation trigger is binary, there is a possibility of entities going back and forth between stages 2 and 1 on a monthly or quarterly basis. To alleviate this, an institution may apply a business rule, such as “stage 2 for at least twelve months”, which would also lead to provisions stability.

However, we believe that, with time, the process will become smooth. Institutions will migrate to forward-looking PIT PD term structures and one-year PIT PDs, and process automation will lead to a more forward-looking early warning system.

6 SUMMARY

In this paper, we focus on the IFRS 9 requirements for significant deterioration. We understand these requirements as (i) defining risk of default, (ii) assessing the definition of risk of default against the requirement, (iii) coming up with a business
logic definition of risk of default customized to an institution’s portfolios and data availability, (iv) defining a trigger mechanism of quantifying significance, (v) defining threshold levels to be used for the quantification of significant change and (vi) devising a robust implementation strategy to run this process in production.

We discuss how the hybrid and TTC-like PDs produced by many legacy, PD and grading models do not properly quantify changes in default risk over time, and so an institution needs to convert its PDs to PIT measures. We then demonstrate some options for evaluating whether significant deterioration has occurred. We offer three metrics for comparing PIT PD term structures.

We discuss the pros and cons of using indicators other than PIT PDs for representing the risk of default, and we show that such measures tend to be backward looking, contrary to the desired forward-looking nature of IFRS 9. Thus, it seems best that those other indicators are limited to an interim implementation or to being used as a backstop.

We explore ways of designing a trigger for stage 2 allocation and recommend a combination of low credit risk and PD multiple and absolute levels. The proper way of determining thresholds is far from settled, but a consensus will surely emerge over time.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

REFERENCES


